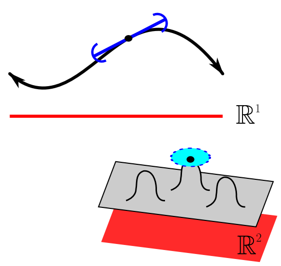
1 Topological Properties (17 March)

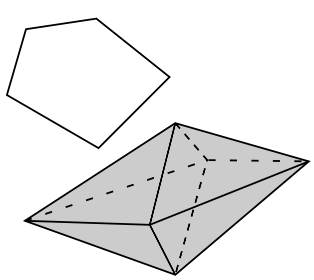
The previous two lectures we defined and gave examples of simplicial complexes. In this lecture, we will look at two invariants of a topological space, specifically: the orientation and the Euler characteristic.

Manifolds. A (geometric) n-manifold $\mathbb{M} \subseteq \mathbb{R}^d$ (for some $d \geq n$) is a closed topological space that resembles Euclidean space at each point. Specifically, that means that each $x \in \mathbb{M}$ has a neighborhood $N_x \subset \mathbb{M}$ that is *homeomorphic* to (read: resembles) an open n-ball $\mathbb{B}_n := \{x \subset \mathbb{R}^n | |x| < 1\}$. [Note: A *closed space* is a space that contains all of its limit/accumulation points.]

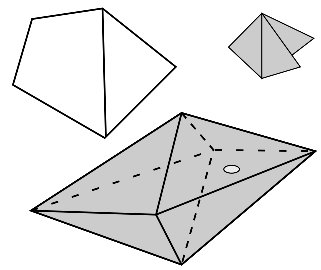
Getting into the definition of homeomorphism is beyond the scope of this lecture, so we will define it via examples, so that you have an intuition for what it is. Recalling tangent lines and planes from calculus class, we remember that a tangent at a point is the line (or plane) that looks close enough to the (smooth) surface/function when you zoom in far enough. For this reason, the plane $\mathbb{R}^d$ as well as the graph any smooth function over $\mathbb{R}^d$:



Allowing for the *tangent* ball above to be a "topological tangent" as opposed to a geometric one, we allow the tangent to bend and change shape, as long as we do not take scissors or glue to the ball. For example, the following spaces are also manifolds:

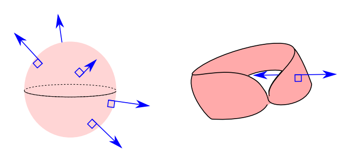


When a space is punctured (a hole added interior to a face), or has an $(n-1)$-dimensional face with three or more n-dimensional simplices adjacent to it, then this space is not a manifold. For example, the following topological spaces are not manifolds:



Note: there is a special group of non-manifolds known as *manifolds with boundary*. These occur if we have a manifold and remove an open disc from it.

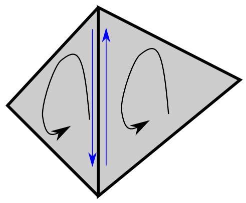
Orientation. When we say that a manifold is *orientable*, that means there is a consistent way of defining *up*. For example, the surface of the earth is orientable. At every point on the earth, we can ask: in which direction is the sky? And, all answers will be locally consistent. A möbius band, however, is not orientable, as a point x can have two different notions of *up*.



Formally, the orientation of a simplex is an ordering of the vertices up to even permutations. The phrase *up to even permutations* means that two permutations are considered equivalent if they differ by an even number of two-element swaps.

For example, if we have a triangle $t = \{abc\}$, then the permutations abc, bca, and cab all represent the same orientation, and cba, acb, and bac all represent a different orientation. Geometrically, this translates to using the *right hand rule* to determine the orientation of a simplex. Whether we obtain the orientation abc or cba depends on whether we are looking at the front or the back of the simplex. A simplex has exactly two orientations, and we often refer to one as the positive $(+1)$ orientation and the other as the negative (-1) orientation.

In a simplicial complex, we say that two adjacent simplices are consistently ordered if the common face is oriented in opposite directions in each simplex. For example, the following triangles are consistently oriented:



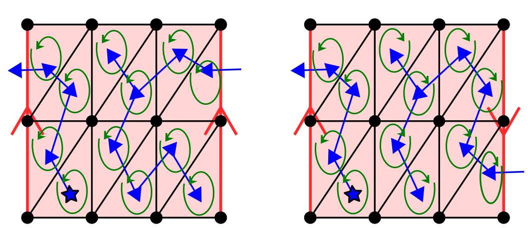
We say that an n-dimensional simplicial complex is *orientable* if all pairs of adjacent n-simplices are consistent.

This definition goes hand-in-hand with algorithm to test if a simplex is orientable or not. Letting K be a k-dimensional simplicial complex (of a manifold) with $k+1$ n-simplices. The algorithm to decide if K is orientable is as follows:

1. Choose a simplex $\sigma \in K$.
2. Let $\tau_1, \ldots, \tau_k$ be the DFS ordering of the n-faces from σ.
3. Choose an arbitrary orientation for σ.
4. $S = \{\sigma\}$.
5. For each $i = 1 \ldots k$: If τ_i has a unique consistent orientation given the orientations of all simplices in S, orient τ_i and set $S = S \cup \{\tau_i\}$. Else, return false.
6. For each edge: If the the incident triangles are not consistently oriented, return false.
7. return true (as we have constructed a consistent orientation).

[Exercise: This is done essentially with two sweeps: doing the DFS and then going through the edges. In fact, we can actually do this in one sweep. Do you see how?]

We consider the (orientable) cylinder and the (non-orientable) Möbius strip. Below, we triangulate the fundamental polygons for each, and attempt to choose an orientation for each triangle, starting with the starred triangle.



Euler Characteristic. Given a planar embedded graph G with v vertices and e edges, let f be the number of pieces the plane is cut into when we remove (cut-along) G. The Euler characteristic of the graph is then: $\chi(G) = v - e + f$. We noticed that $\chi(G) = 2$ always. This is actually a specific example of the Euler characteristic of the sphere. A graph is planar if and only if it can be embedded on the surface of a sphere. No matter how you embed it on the sphere (or on the plane), the number of *faces* the embedding creates will always be the same.

Now consider a simplicial complex K. Let K_i be the set of i-simplices in K. Then, the Euler characteristic of K is defined as:

$$\chi(K) := \sum_{i=0}^{\infty} (-1)^i |K_i|.$$

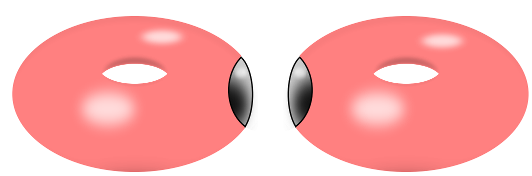
If we use colloquial language, we see that the Euler characteristic is equal to the number of vertices minus the number of edges plus the number of faces minus the number of tetrahedra plus the number of 4-simplices, etc.

The Euler characteristic is a topological invariant. This means, if we have the same underlying space, the Euler characteristic will always be the same, no matter how we triangulate it.

Manifold Classification.

CLASSIFICATION THEOREM. Given an orientable compact manifold, the Euler characteristic is sufficient to uniquely determine the topological type of the manifold: the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$, the double torus $\mathbb{T}^2 \# \mathbb{T}^2$, the connected sum of three tori $\mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{T}^2$, etc.

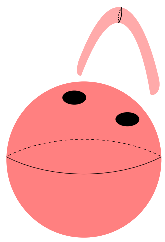
The connected sum $A \# B$ is obtained by removing an open disc from both A and B, then gluing them together along the boundaries of the removed discs. This operation is one type of *surgery* in topology.



In fact, all non-orientable two-manifolds can also be classified by their Euler characteristic. A non-orientable two-manifold is either a projective plane $\mathbb{P}^2$ or the connected sum of projective planes $\mathbb{P}^2 \# \cdots \# \mathbb{P}^2$.

Handle-body Decomposition. Note: the handle-body decomposition was not explicitly discussed in class, but it might be helpful to understand for Question 4 of the homework.

The torus $\mathbb{T}^2$ can be obtained by starting with a sphere, removing two holes, and connecting the holes using a cylinder. This cylinder we call a *handle*. We can repeat this process to add n handles, we obtain the connected sum of n tori.



Summary. Given two simplicial complexes K_1 and K_2, it is often very difficult to determine if $K_1 = K_2$, or if K_1 is topologically equivalent to K_2. Instead, we choose a set or properties to describe K_1 and K_2. If these properties are topological invariants, then we can prove that $K_1 \neq K_2$ by finding a property that witnesses a difference between them (for example, perhaps K_1 is orientable and K_2 is not). More often than not, we do not even need to fully understand exactly what the complex K_1 represents, we may just be interested in the properties themselves: determining whether K_1 is connected may be sufficient for some applications.