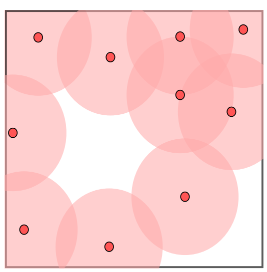
1 Point Cloud Data (12 March)

In the last class, we defined simplicial complexes. Today, we will show how we can construct simpicial complexes from point cloud data. We started class by thinking of ourselves as data points, creating complexes to represent the sturcture when the data is our names (we formed a line, by organizing alphabetically) or our month and day of birth (we formed a circle).

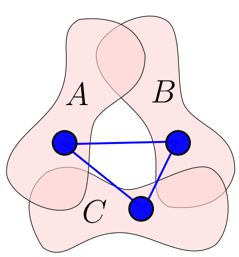
Network Coverage. We begin with a motivating example: network coverage. Suppose we have n sensors in $D \subset \mathbb{R}^d$, each with coverage radius r. We will assume that D is compact (i.e., closed and bounded). For example, consider cell towers. We want to know where, if at all, are there holes in the network coverage? For example, consider the following set of ten sensors in the square region below:



Nerve Complex. Let F be a set of sets. We define the Nerve of F as follows:

$$\mathsf{Nrv}(F) := \{X \subset F | \bigcap_{X_i \in X} X_i \neq \emptyset\}.$$

As an example, consider the following set $F = \{A, B, C\}$ (in pink) and $\mathsf{Nrv}(F)$ (in blue): In this example, the Nerve is given as follows:

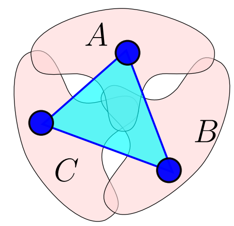


$$\mathsf{Nrv}(F) = \{A, B, C, AB, AC, BC\}.$$

(Note: for ease of exposition, we abuse notation slightly and write AB for the set $\{A, B\}$). The set ABC is not in $\mathsf{Nrv}(F)$, since $A \cap B \cap C = \emptyset$. Recalling the definition of an abstract simplicial complex, one can show:

LEMMA. The Nerve complex is an abstract simplicial complex.

In general, the Nerve complex does not have the same *topology* as F. Here is an example of where the Nerve complex fails to capture the structure of F: Notice that the union of the sets in F has three holes; whereas, the complex $\mathsf{Nrv}(F)$ has no holes. (We will formalize the definition of a hole in the next lecture). However, under the right assumptions, the Nerve complex does capture the right topology:



(WEAK) NERVE THEOREM. If F is a set of convex sets, then $\bigcup F$ and $\mathsf{Nrv}(F)$ have the same homotopy type.

There is a stronger version of this theorem that holds for more general assumptions on F, but we will focus on the case where F is a set of balls (or discs).

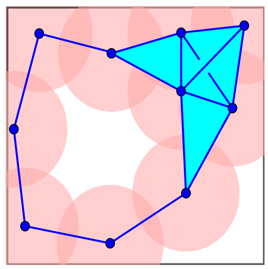
NOTE: If two topological spaces have the same homotopy type, this is a very strong statement. In not-so-technical-terms, we can say that the two spaces have the same types of and numbers of holes. `http://commons.wikimedia.org/wiki/File:Mug_and_Torus_morph.gif`

Čech Complex The nerve of a set of balls cnetered at points $S \subset \mathbb{R}^d$ is called the Čech complex:

$$\check{\mathsf{C}}\mathsf{ech}(S, r) = \mathsf{Nrv}(B = \{B_r(s)\}_{s \in S}).$$

We make two observations: First, since the sets in B are all convex, the Nerve theorem applies. Second, we notice that this complex can grow rather large. Given n points, the complex can grow as large as the complete complex on n points. [Exercise: how many simplices would that be?]

A further (related) complication is the fact that if S is in $\mathbb{R}^d$, Čech(S,r) may not be embeddable in $\mathbb{R}^d$. As an example, consider our network coverage problem from the begining of the lecture. The points are given in $\mathbb{R}^2$, but the Čech complex contains a tetrahedron (on the top-right):



Alpha Complex To alleviate the dimensionality issue that arises with the Čech complex, we define the Alpha complex, which is homotopic to the Čech complex, but (typically) much fewer simplices.

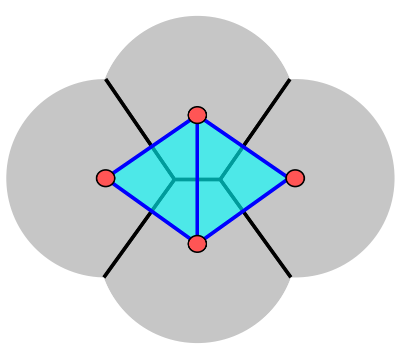
Let $S \subset \mathbb{R}^2$ be a finite point set. For each $s \in S$, let V_s be the Voronoi cell associated to s. Letting

$$R = \{B_r(s) \cap V_s\}_{s \in S},$$

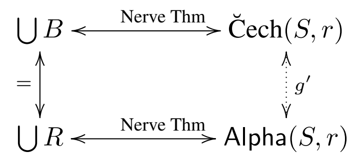
we define the Alpha complex as follows:

$$Alpha(S,r) = \mathsf{Nrv}(R).$$

The worst case complexity of the Alpha complex is the size of the Delaunay triangulation, not the complete complex on all sites. We demonstrate the Alpha complex using four sites whose Čech complex is the tetrahedron: The



Čech complex has 1 tetrahedron, 4 triangles, 6 edges, and four vertices, and can be embedded in $\mathbb{R}^3$ but not $\mathbb{R}^2$. The Alpha complex has 2 triangles, 5 edges, and 4 vertices, and can be embedded in $\mathbb{R}^2$. However, as we observe in the diagram below, both the Čech complex and the Alpha complex have the same homotopy type (i.e., there is a bijection between the holes).



Summary. Today, we discussed different ways to go from point cloud data to structred objects that we can study with a computer: simplicial complexes. There are other ways to construct simplicial complexes from data (for example, the Vietoris-Rips complex, using non-Euclidean metrics, ...). The important thing is that we can use these discrete objects (simplicial complexes) in order to study the structure of point cloud data.