1 Simplicial Complexes (10 March)

Today's lecture began with a power point slide highlighting some applications of topology. The main point of these examples is that, sometimes, geometry alone doesn't tell the whole story. Having a *circle* is really an unattainable goal when working with data ... but, we can instead look for properties of the data that are circle-like (e.g., does the shape create an *inside* and an *outside*?).

Simplicial Complexes are one of the standard tools that computational topologists use. This lecture will provide a definition of simplicial complexes, as well as give several examples so that we can get our hands dirty.

Graphs. We will start with something that you should be very familiar with: graphs. Today, we formalized the definition: A *graph* G is a vertex set V along with an edge set $E \subset V \times V$. We may write $G = (V, E)$ without explicitly saying that V is a vertex set and E is the edge set.

For example, a social network (Facebook) graph can be created by having a vertex for each person and an edge connecting to vertices if the two corresponding people are Facebook friends. What are the questions that we can ask of such a graph?

1. For each friendship, how long has it existed?
2. Who are mutual friends of person A and person B?
3. Is there a *central* person in this network?

To answer these questions, we may need to add weights to the edges (perhaps representing the strength of that friendship) and make definitions. For example, one definition of the central person could be the person p defined as follows:

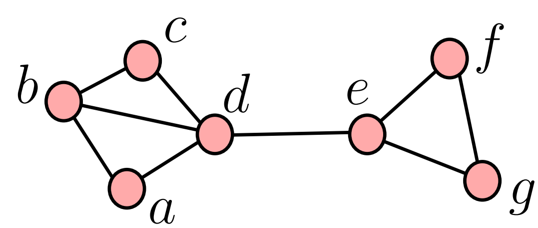
$$p = argmin_{p \in V} \max_{q \neq p \in V} d(q, p),$$

where $d(p, q)$ is the (weighted) distance between p and q in the graph.

Mutual friendship brings up a concept called the clique, which has two definitions:

clique: (real-life) an exclusive group of people sharing a common interest
(graph-theory) a subset $W \subset V$ such that $W \times W \subseteq E$

By convention, we will assume that W must be non-empty to form a clique. So, every (non-empty) graph has at least one clique, as every singleton set forms a clique. Likewise, every edge represents a clique of the two vertices that are its endpoints. The following graph has five cliques of size three or more: {abc}, {bcd}, {abd}, {abcd}, and {efg}.

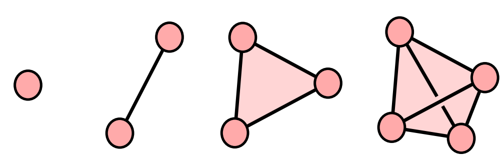


Affine Independence. Let $V = \{v_0, \dots, v_k\} \subset R^d$. We say that the vertices in V are *affinely independent* if the set $\{u_i = v_i - v_0\}_{i \neq j}$ are *linearly independent*. Here, v_0 is acting like the origin, and the vectors u_i are basis vectors. To be linearly independent means that no u_j is a linear combination of $\{u_i\}_{i \neq j}$, i.e., there does not exist $u_j \neq 0$ and coefficients c_i such that $u_j = \sum_{i \neq j} c_i u_i$.

[[Exercise: show that the set $\{u_i\}$ are linearly independent if and only if the set $\{w_j = v_j - u_n\}_{j \neq n}$ are linearly independent for an arbitrary choice of n.]]

Simplices. We can define simplices in two ones: geometrically and abstractly. A *geometric k-simplex σ* is the convex hull of $k + 1$ affinely independent points in Euclidean space; we write $\sigma = \mathsf{CH}(V)$, where V is a set of $k + 1$ affinely independent points in R^d.

Low-dimensional simplices are already familiar objects: zero-simplices are points (vertices), one-simplices are edges, two-simplices are triangles, and three-simplices are tetrahedron (or triangular pyramids). Notice that the



dimension k of a k-simplex refers to the dimension of the *interior* of the simplex. Also note that three co-linear points form a degenerate triangle, which is not a (geometric) simplex, as the three points are not affinely independent.

If σ is a simplex defined by the vertices in V and $W \subset V$ and $\tau = \mathsf{CH}(W)$, we say that τ is a *face* or a *subsimplex* of σ, and we denote this relationship by $\tau < \sigma$. We say that τ is a *proper face* if $\tau \neq \sigma$.

We can also define a simplex abstractly, without the need of an embedding space. Given a set of abstract vertices $V = \{v_0, v_1, \ldots, v_k\}$, an *abstract k-simplex* is a subset of $n + 1$ distinct vertices in V. An k-simplex always has a geometric realization in R^d for all $d \geq n$.

Geometric Simplicial Complexes. A set of simplices K is a (geometric) simplicial complex if it is closed under the operation of taking intersections. What this means is that K satisfies the following two properties:

1. If $\sigma \in K$ and $\tau < \sigma$, then $\tau \in K$.
2. If $\sigma_0, \sigma_1 \in K$, then $\sigma_{01} = \sigma_0 \cap \sigma_1$ is either empty or in K.

The dimension of a simplicial complex K is the maximum dimension of any simplex comprising K.

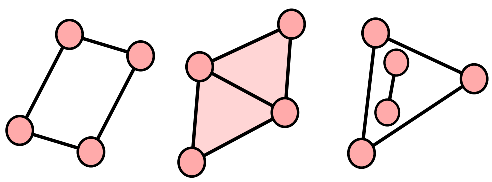
NOTE: A simplicial complex is a set, not a multi-set. So, an simplex (such as an edge) either exists in the set or does not exist. There cannot be two copies of the same exact edge.

Abstract Simplicial Complex. An abstract simplicial complex is a finite collection of sets A such that

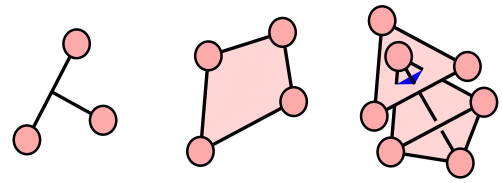
1. If $\alpha \in A$ and $\beta \subseteq \alpha$, then $\beta \in A$.
2. If $\alpha_0, \alpha_1 \in A$, then $\alpha_{01} = \alpha_0 \cap \alpha_1$ is either empty or in A.

The dimension of A is equal to the cardinaltiy of A (i.e., how many elements are in A). If $\beta \subset \alpha$, then β is called a face of α. Just as before, β is a proper face if $\beta \neq \alpha$.

Examples. The following complexes are valid simplicial complexes:



The following complexes are not valid simplicial complexes:



[[Exercise: For each of the s.c. in the previous two sets, why is the s.c. valid or invalid? We discussed this in class.]]

Summary. During today's class, we learned about the building block of computational topology: simplicial complexes. We used examples in order to better understand the unfamiliar definitions. If you feel like you still do not understand the definitions, try to draw some more examples. Getting your hands dirty by constructing examples (and counter-examples) using these definitions is the best way to learn them. While we did not discuss the abstract simplicial complex in this lecture specifically, it is included in these notes and will be the starting point for the next lecture.