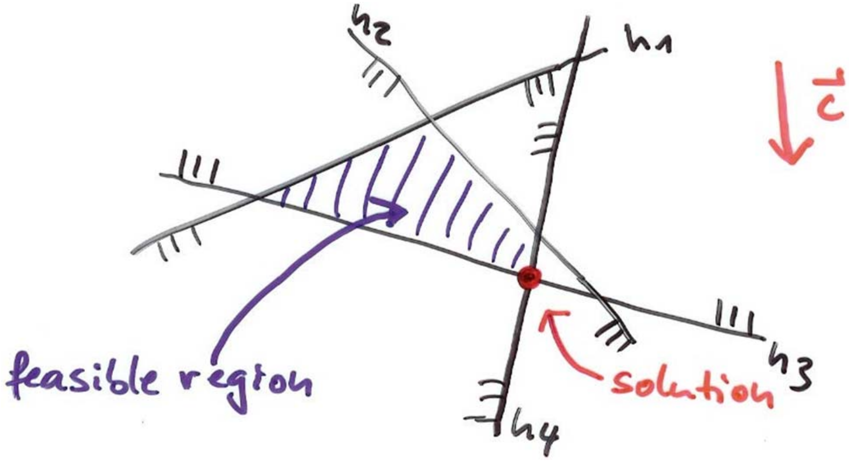
CMPS 3130/6130 Computational Geometry
Spring 2015



Linear Programming and Halfplane Intersection

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Word Problem

A company produces tables and chairs. The profit for a chair is \$2, and for a table \$4. Machine group A needs 4 hours to produce a chair, and 6 hours to produce a table. Machine group B needs 2 hours to produce a chair, and 6 hours to produce a table. Per day there are at most 120 working hours for group A and at most 72 hours for group B.

How can the company maximize profit?

Variables:

c_A = # chairs produced on machine group A
c_B = # chairs produced on machine group B
t_A = # tables produced on machine group A
t_B = # tables produced on machine group B

Constraints:

$4c_A + 6t_A \leq 120$

$2c_B + 6t_B \leq 72$

Objective function (profit):

Maximize $2(c_A+c_B)+4(t_A+t_B)$

Linear Programming

Variables: $x_1,\ldots,x_d$

Objective function:
Maximize $f_{\vec{c}}(\vec{x}) = c_1 x_1+\ldots+c_d x_d$

Constraints:

h_1: $a_{11} x_1+\ldots+a_{1d} x_d \leq b_1$

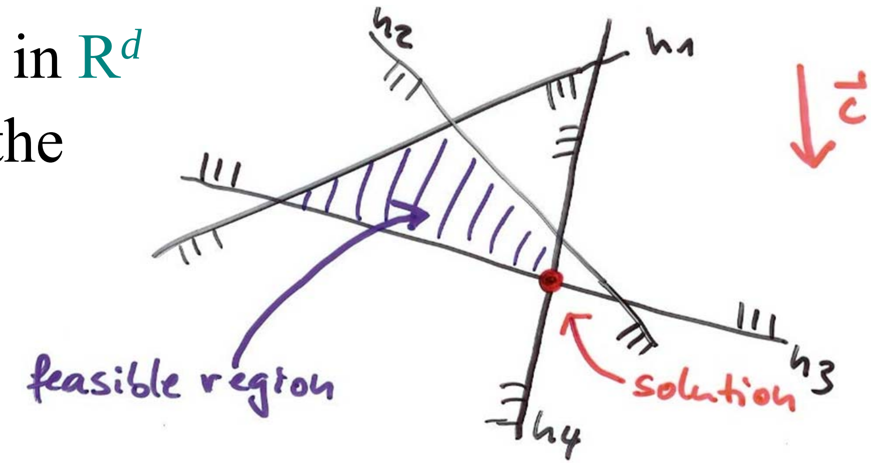
h_2: $a_{21} x_1+\ldots+a_{2d} x_d \leq b_2$

. . .

h_n: $a_{n1} x_1+\ldots+a_{nd} x_d \leq b_n$

Linear program in d variables with n constraints

- Each constraint h_i is a half-space in $\mathbb{R}^d$
- $\bigcap_{i=1}^{n} h_i$ is the feasible region of the linear program
- Maximizing $f_{\vec{c}}(\vec{x})$ corresponds to finding a point $\vec{x}$ that is extreme in direction $\vec{c}$.



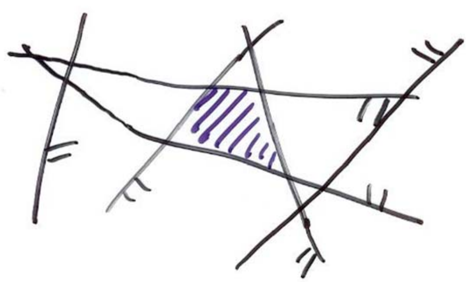
Sub-Problem: Halfspace Intersection (in R^2: Halfplane Intersection)

Given: A set $H=\{h_1, h_2, \ldots, h_n\}$ of halfplanes

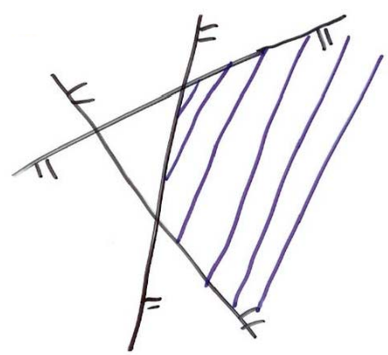
$$h_i: \qquad a_i x + b_i y \leq c_i$$

with constants a_i, b_i, c_i ; for $i=1,\ldots,n$.

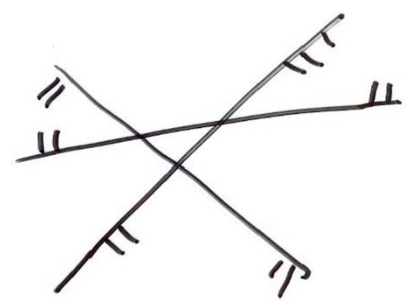
Find: $\bigcap_{i=1}^{n} h_i$, i.e., the feasible region of all points $(x,y)\in R^2$ satisfying all n constraints at the same time. This is a convex polygonal region bounded by at most n edges.



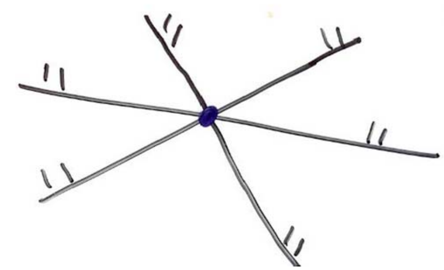
intersection bounded



intersection unbounded



intersection empty



intersection degenerated to a point

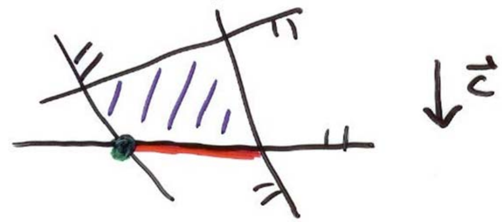
D&C Halfplane Intersection

Algorithm Intersect_Halfplanes(H):
Input: A set H of n halfplanes in $\mathbf{R}^2$
Output: The convex polygonal region $C = \bigcap_{h \in H} h$
if $|H|=1$ then
    $C = h$, where $H=\{h\}$
else
    split H into two sets H_1 and H_2 of size $n/2$ each
    C_1 = Intersect_Halfplanes(H_1)
    C_2 = Intersect_Halfplanes(H_2)
    C = Intersect_Convex_Regions(C_1, C_2)
    return C

- Use a plane-sweep to develop an $O(n)$-time algorithm for Intersect_Convex_Regions
- $T(n) = 2T(n/2)+n \quad \Rightarrow \quad T(n) \in O(n \log n)$

Incremental Linear Programming

- 2D linear program (LP)
- Assume the LP is bounded (otherwise add constraints)
- Assume there is one unique solution (if any); take the lexicographically smallest solution



- **Incremental approach:** Add one halfplane after the other.

$H_i = \{h_1, \dots, h_i\}$ $\quad C_i = h_1 \cap \dots \cap h_i$ $\quad C = C_n = \bigcap_{h \in H} h$

Let v_i = unique optimal vertex for feasible region C_i, for $i \geq 2$.

Then $C_1 \supseteq C_2 \supseteq \dots \supseteq C_n = C$, and hence
if $C_i = \emptyset$ for some i then $C_j = \emptyset$ for all $j \geq i$.

Incremental Linear Programming

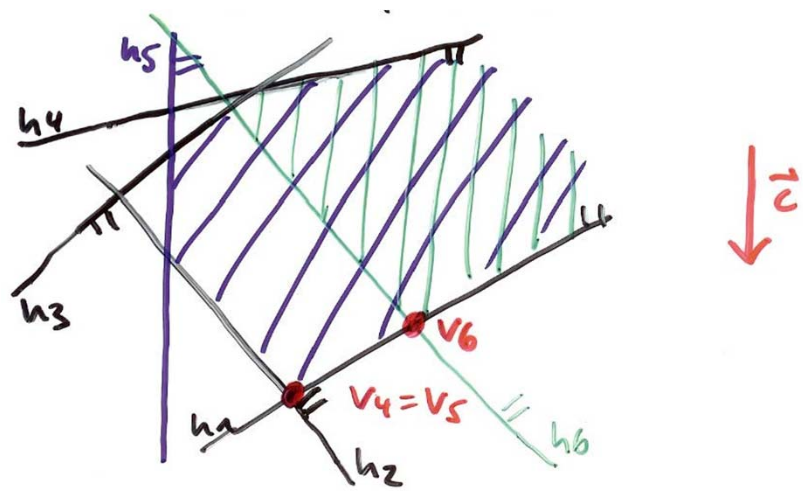
Lemma: Let $2 \le i \le n$.

(i) If $v_{i-1} \in h_i$ then $v_i = v_{i-1}$

(ii) If $v_{i-1} \notin h_i$ then

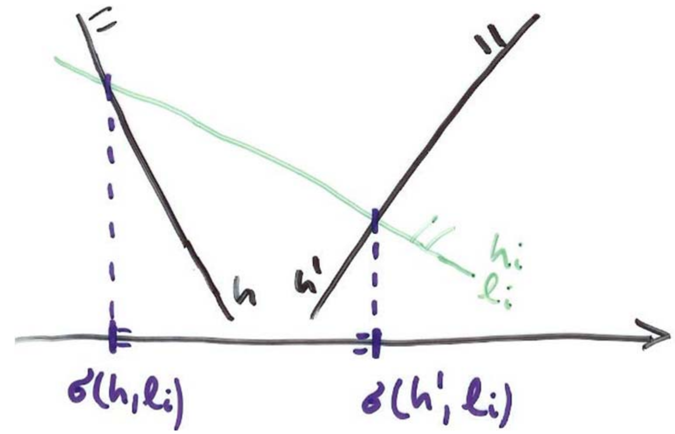
$C_i = \emptyset$

or $v_i \in l_i$ = the line bounding h_i



Handling case (ii) involves solving a 1-dimensional LP on l_i :

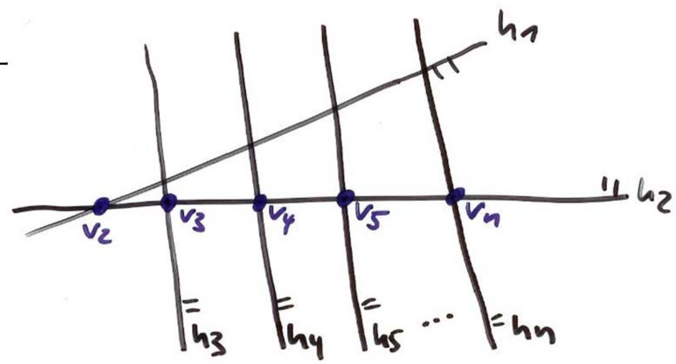
- The feasible region is just an interval, that can be computed in linear time [rightmost left-bounded halfplane, leftmost right-bounded halfplane]
- $\Rightarrow$ We can compute a new v_i, or decide that the LP is infeasible, in $O(i)$ time



2D_Bounded_LP

Algorithm 2D_Bounded_LP(H , $\vec{c}$):
Input: A two-dimensional LP (H , $\vec{c}$)
Output: Report if (H , $\vec{c}$) is infeasible. Otherwise report the lexicographically smallest point that maximizes $f_{\vec{c}}$.
Let $h_1, \ldots, h_n$ be the halfplanes of H
Let v_2 be the corner of C_2, which exists because LP is bounded
for i=3 to n do
    if $v_{i-1} \in h_i$ then $v_i = v_{i-1}$
    else // Case (ii)
        v_i = point on l_i that maximizes $f_{\vec{c}}$ subject to constraints in H_{i-1}
        if such a point does not exist then
            Report that the LP is infeasible
            break;
return v_n

- Runtime: $\sum_{i=1}^{n} O(i) = O(n^2)$
  Storage: $O(n)$



Randomized Incremental LP

Depending on the insertion order of the halfplanes the runtime varies between $O(n)$ and $O(n^2)$.
$\Rightarrow$ Randomize the input order of the halfplanes.

Theorem: 2D_Randomized_Bounded_LP runs in $O(n)$ expected time and $O(n)$ deterministic space.

Proof: Define a random variable $X_i = \begin{cases} 1, v_{i-1} \notin h_i \\ 0, else \end{cases}$

The total time spent to resolve case (ii), over all $h_1, \ldots, h_n$ is

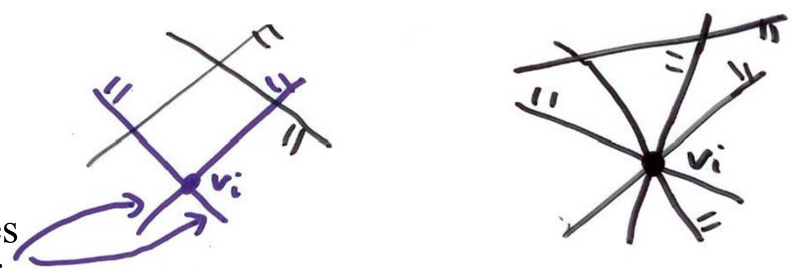
$$\sum_{i=1}^{n} O(i) X_i$$

Randomized Incremental LP

We now need to bound the expected value
$E(\sum_{i=1}^{n} O(i)X_i) = \sum_{i=1}^{n} O(i)E(X_i)$
and we know that $E(X_i) = P(X_i) = P(v_{i-1} \notin h_i)$.

Apply backwards analysis to bound $E(X_i)$:

- Fix $H_i = \{h_1, \ldots, h_i\}$ which determines C_i.
- Analyze what happened in last step when h_i was added.
- P(had to compute new optimal vertex when adding h_i)
  = P(optimal vertex changes when we remove a halfplane from C_i)
  $\leq \frac{2}{i}$



2 lines
defining v_i

$\Rightarrow E(X_i) \leq \frac{2}{i}$

$\Rightarrow$ Total expected runtime is $\sum_{i=1}^{n} O(i)\frac{2}{i} = O(n)$ □