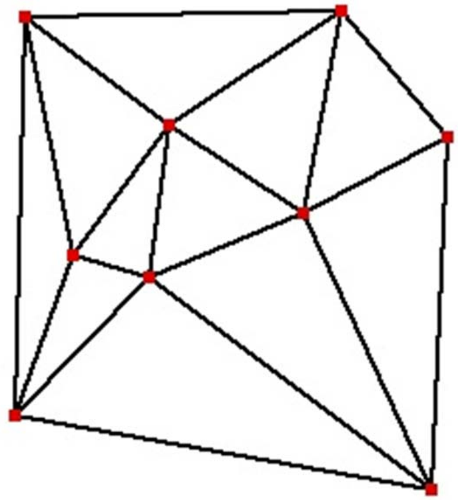
CMPS 3130/6130 Computational Geometry
Spring 2015



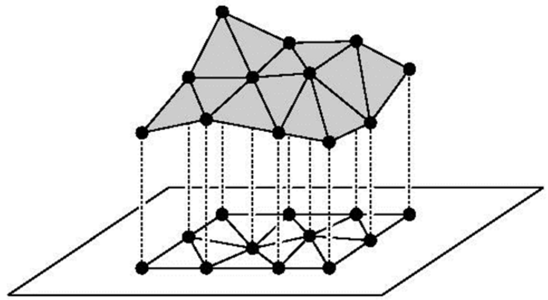
Delaunay Triangulations II

Carola Wenk

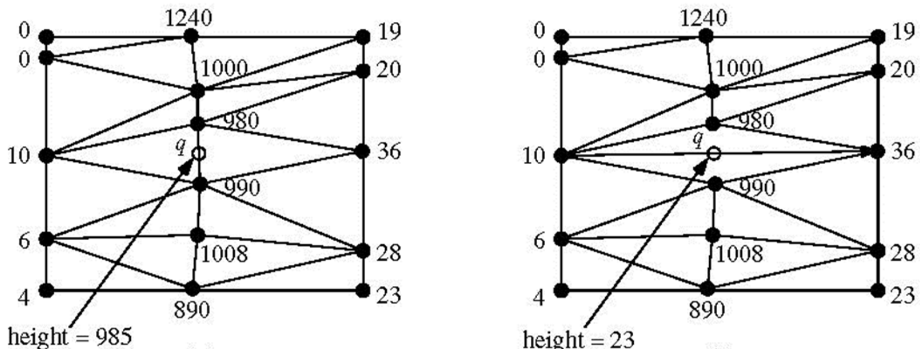
Based on:

Computational Geometry: Algorithms and Applications

Applications of DT



- Terrain modeling:
  - Model a scanned terrain surface by interpolating the height using a piecewise linear function over R^2.
  - Angle-optimal triangulations give better approximations / interpolations since they avoid skinny triangles

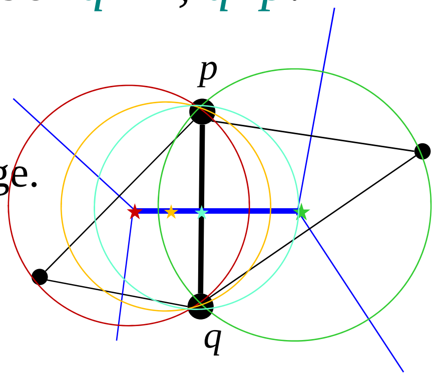


(a) (b)

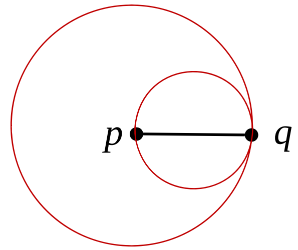
Applications of DT

- All nearest neighbors: Find for each $p \in P$ its nearest neighbor $q \in P$; $q \neq p$.

  – **Empty circle property:** $p,q \in P$ are connected by an edge in DT(P) $\Leftrightarrow$ there exists an empty circle passing through p and p.
  Proof: "$\Rightarrow$": For the Delaunay edge pq there must be a Voronoi edge. Center a circle through p and q at any point on the Voronoi edge, this circle must be empty.
  "$\Leftarrow$": If there is an empty circle through p and q, then its center c has to lie on the Voronoi edge because it is equidistant to p and q and there is no site closer to c.



  – **Claim:** Every $p \in P$ is adjacent in DT(P) to its nearest neighbor $q \in P$.
  Proof: The circle centered at p with q on its boundary has to be empty, so the circle with diameter pq is empty and pq is a Delaunay edge.

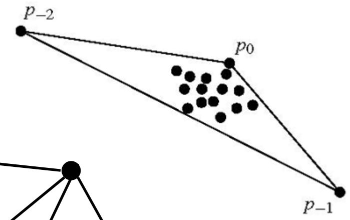


  – **Algorithm:** Find all nearest neighbors in O(n) time: Check for each $p \in P$ all points connected to p with a Delaunay edge.

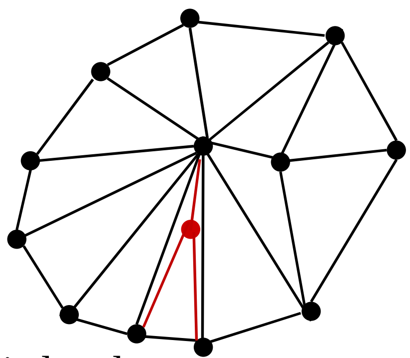
- Minimum spanning tree: The edges of every Euclidean minimum spanning tree of P are a subset of the edges of DT(P).

Randomized Incremental Construction of DT(P)

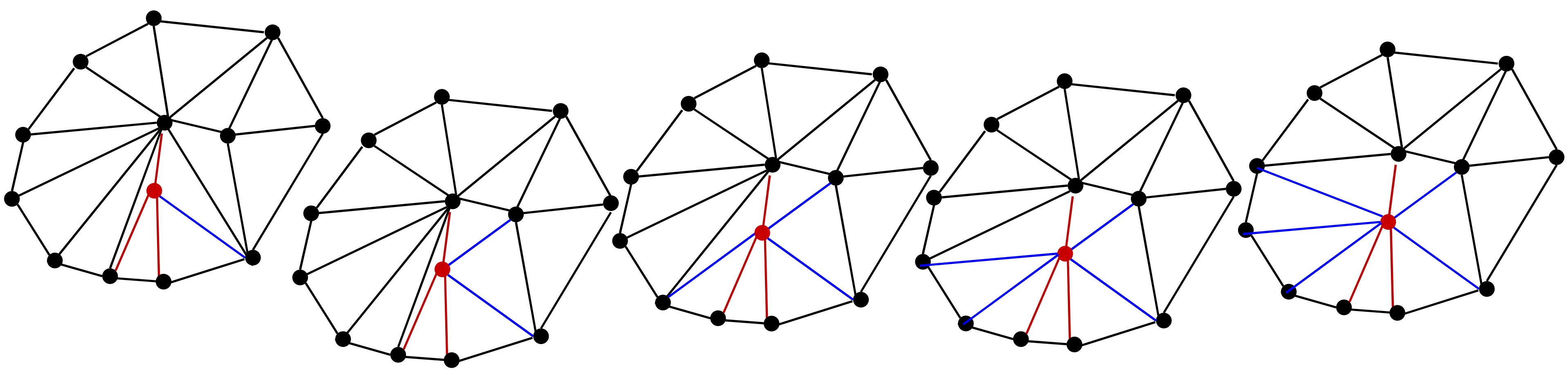
- Start with a large triangle containing *P*.



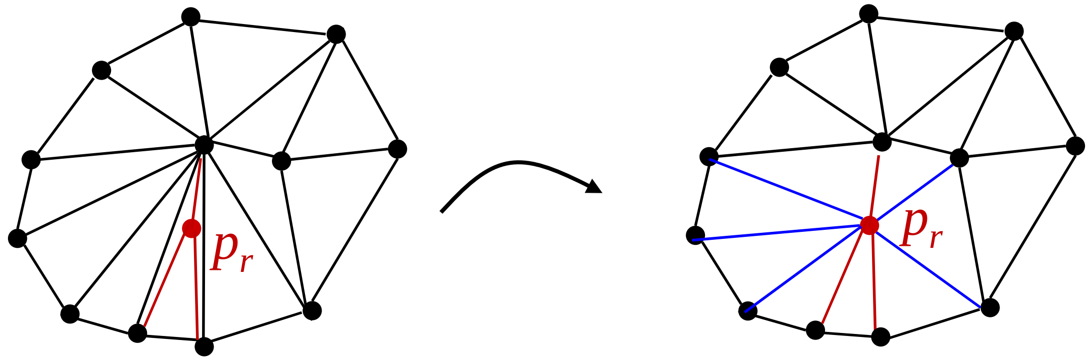
- Insert points of *P* incrementally:
  - Find the containing triangle
  - Add new edges



  - Flip all illegal edges until every edge is legal.



Randomized Incremental Construction of DT(P)



- An edge can become illegal only if one of its incident triangles changes.
- Check only edges of new triangles.
- Every new edge created is incident to p_r.
- Every old edge is legal (if p_r is on on one of the incident triangles, the edge would have been flipped if it were illegal).
- Every new edge is legal (since it has been created from flipping a legal edge).

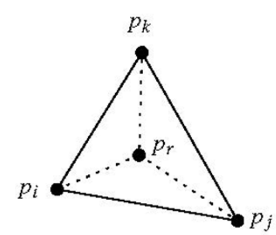
Pseudo Code

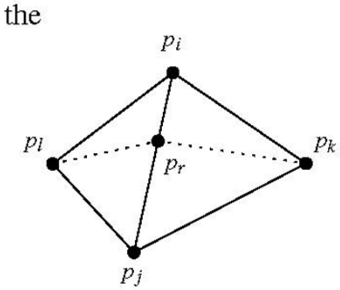
Algorithm DELAUNAYTRIANGULATION(P)
Input. A set P of $n+1$ points in the plane.
Output. A Delaunay triangulation of P.

1. Let p_0 be the lexicographically highest point of P, that is, the rightmost among the points with largest y-coordinate.
2. Let p_{-1} and p_{-2} be two points in $\mathbb{R}^2$ sufficiently far away and such that P is contained in the triangle $p_0 p_{-1} p_{-2}$.
3. Initialize $\mathcal{T}$ as the triangulation consisting of the single triangle $p_0 p_{-1} p_{-2}$.
4. Compute a random permutation $p_1, p_2, \dots, p_n$ of $P \setminus \{p_0\}$.
5. **for** $r \leftarrow 1$ **to** n
6.   **do** (∗ Insert p_r into $\mathcal{T}$: ∗)
7.   Find a triangle $p_i p_j p_k \in \mathcal{T}$ containing p_r.
8.   **if** p_r lies in the interior of the triangle $p_i p_j p_k$
9.   **then** Add edges from p_r to the three vertices of $p_i p_j p_k$, thereby splitting $p_i p_j p_k$ into three triangles.
10.   LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)
11.   LEGALIZEEDGE($p_r, \overline{p_j p_k}, \mathcal{T}$)
12.   LEGALIZEEDGE($p_r, \overline{p_k p_i}, \mathcal{T}$)
13.   **else** (∗ p_r lies on an edge of $p_i p_j p_k$, say the edge $\overline{p_i p_j}$ ∗)
14.   Add edges from p_r to p_k and to the third vertex p_l of the other triangle that is incident to $\overline{p_i p_j}$, thereby splitting the two triangles incident to $\overline{p_i p_j}$ into four triangles.
15.   LEGALIZEEDGE($p_r, \overline{p_i p_l}, \mathcal{T}$)
16.   LEGALIZEEDGE($p_r, \overline{p_l p_j}, \mathcal{T}$)
17.   LEGALIZEEDGE($p_r, \overline{p_j p_k}, \mathcal{T}$)
18.   LEGALIZEEDGE($p_r, \overline{p_k p_i}, \mathcal{T}$)
19. Discard p_{-1} and p_{-2} with all their incident edges from $\mathcal{T}$.
20. **return** $\mathcal{T}$

LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)

1. (∗ The point being inserted is p_r, and $\overline{p_i p_j}$ is the edge of $\mathcal{T}$ that may need to be flipped. ∗)
2. **if** $\overline{p_i p_j}$ is illegal
3.   **then** Let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
4.   (∗ Flip $\overline{p_i p_j}$: ∗) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
5.   LEGALIZEEDGE($p_r, \overline{p_i p_k}, \mathcal{T}$)
6.   LEGALIZEEDGE($p_r, \overline{p_k p_j}, \mathcal{T}$)

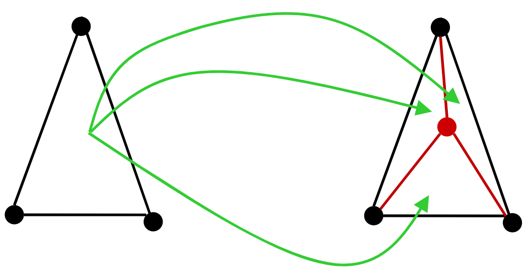




History

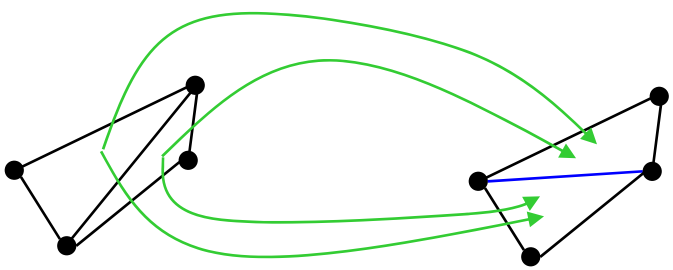
The algorithm stores the history of the constructed triangles. This allows to easily locate the triangle containing a new point by following pointers.

- Division of a triangle:



Store pointers from the old triangle to the three new triangles.

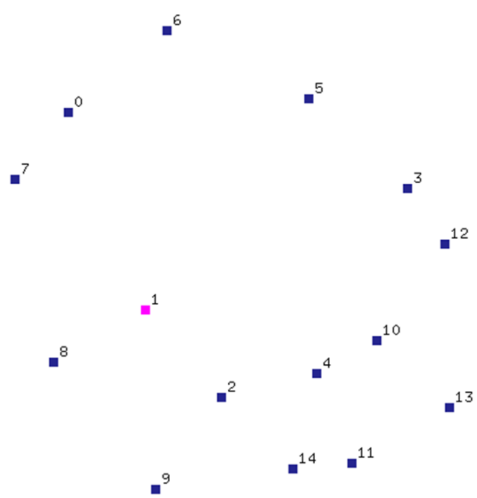
- Flip:

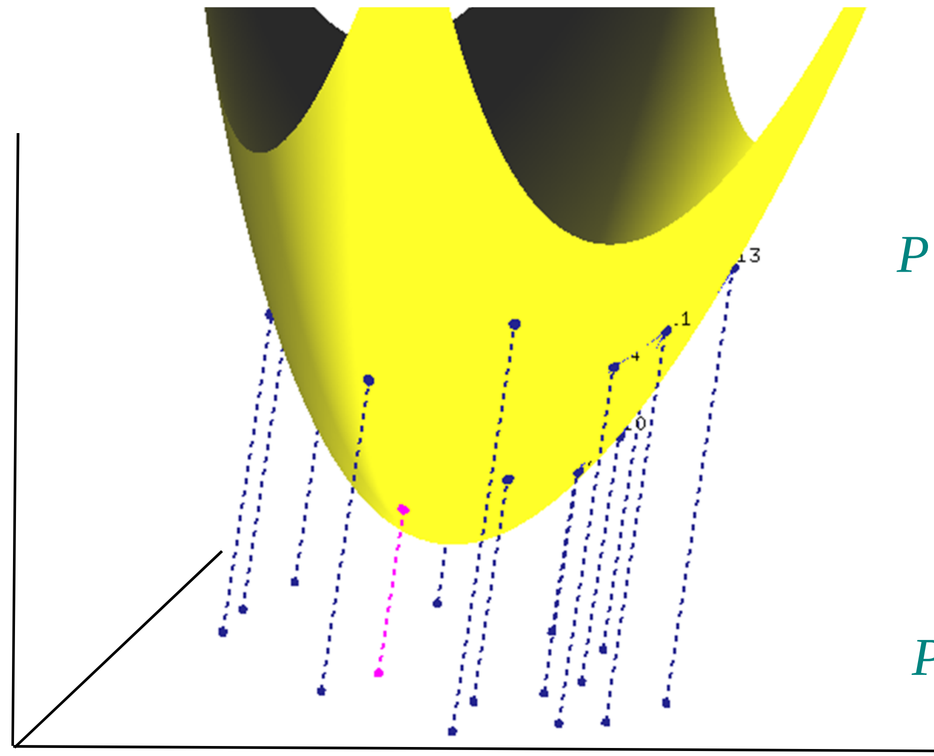


Store pointers from both old triangles to both new triangles.

DT and 3D CH

Theorem: Let $P=\{p_1,\ldots,p_n\}$ with $p_i=(a_i, b_i, 0)$. Let $p^*_i = (a_i, b_i, a^2_i + b^2_i)$ be the vertical projection of each point p_i onto the paraboloid $z=x^2+y^2$. Then $DT(P)$ is the orthogonal projection onto the plane $z=0$ of the lower convex hull of $P^*=\{p^*_1,\ldots,p^*_n\}$.





P^*

P

Pictures generated with Hull2VD tool available at http://www.cs.mtu.edu/~shene/NSF-2/DM2-BETA

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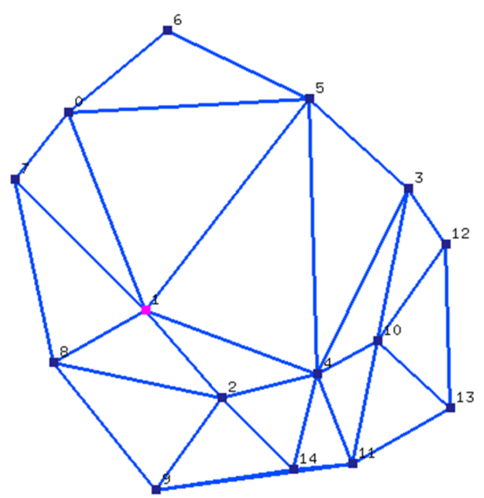

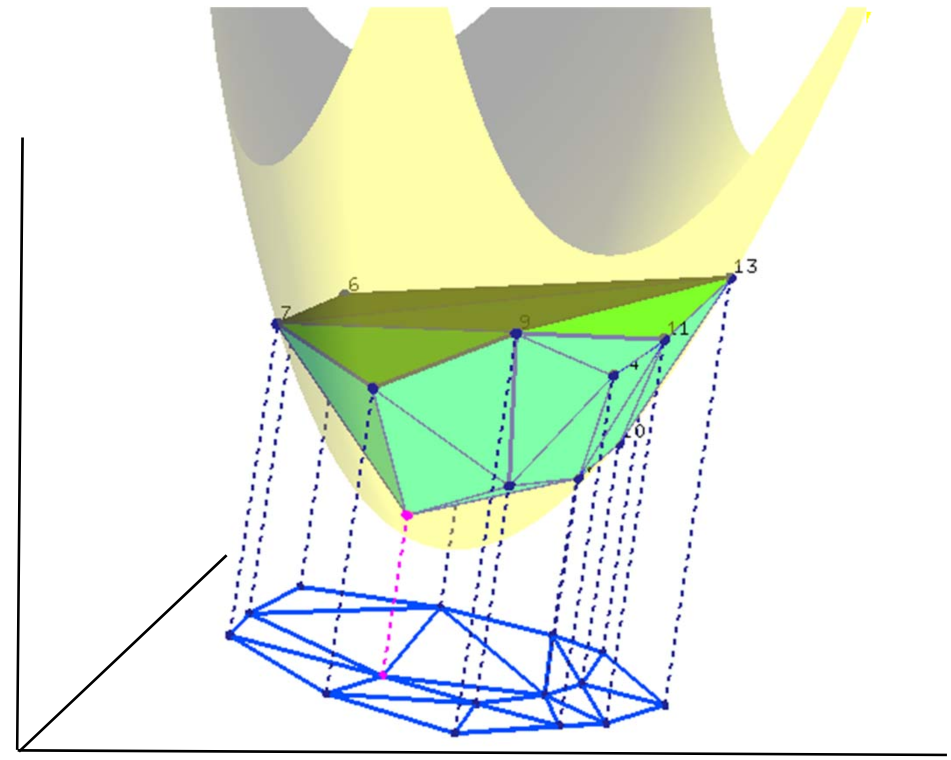

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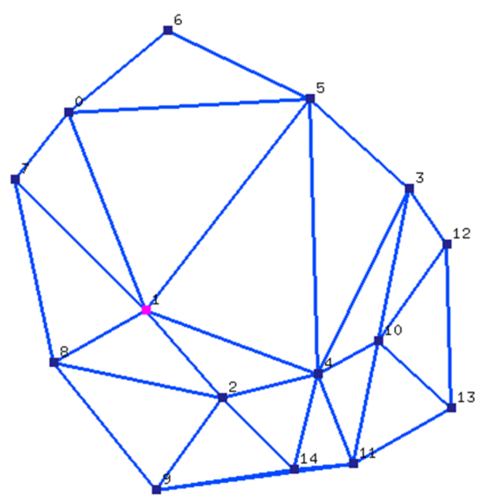

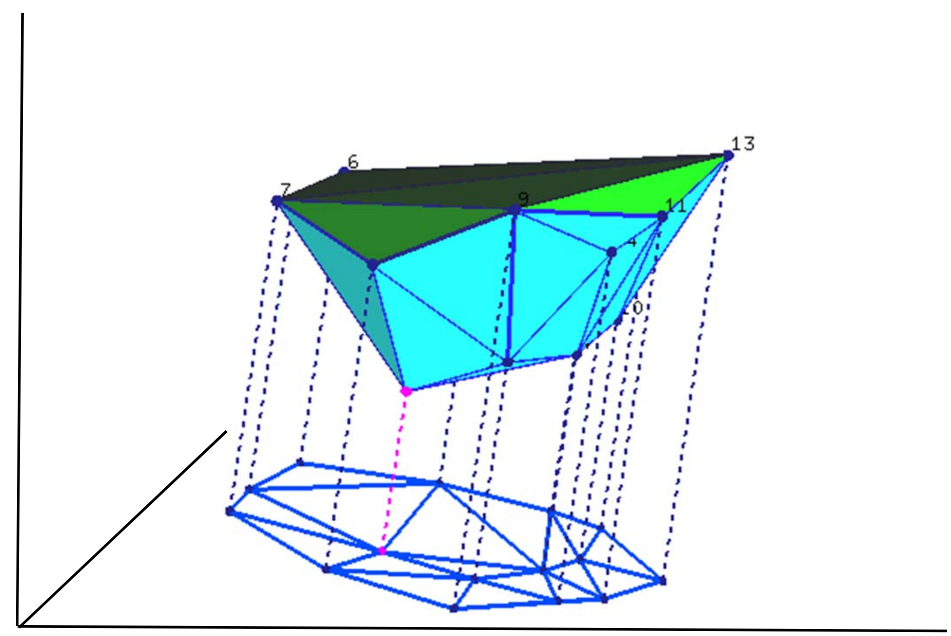

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DT and 3D CH

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p^*_i, p^*_j, p^*_k form a (triangular) face of LCH(P^*)

$\Leftrightarrow$

The plane through p^*_i, p^*_j, p^*_k leaves all remaining points of P above it

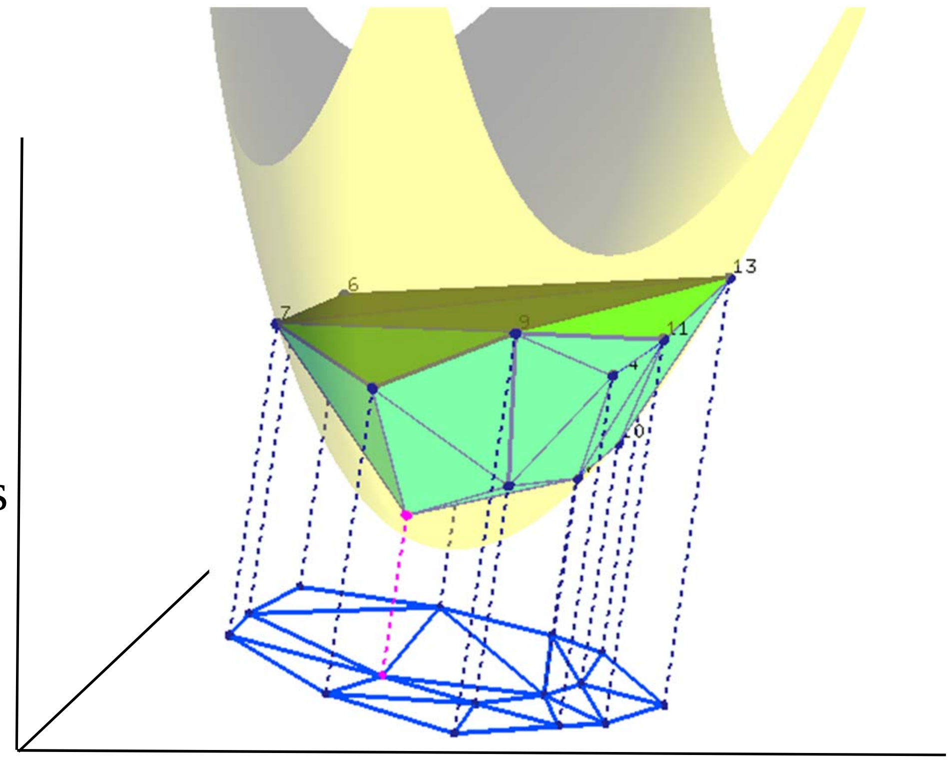
property of unit paraboloid

$\Leftrightarrow$

The circle through p_i, p_j, p_k leaves all remaining points of P in its exterior

$\Leftrightarrow$

p_i, p_j, p_k form a triangle of DT(P)



Slide adapted from slides by Vera Sacristan.