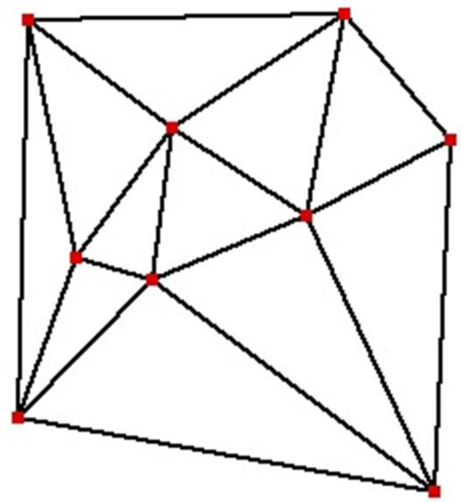
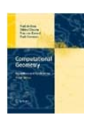
CMPS 3130/6130 Computational Geometry
Spring 2015



Delaunay Triangulations I

Carola Wenk

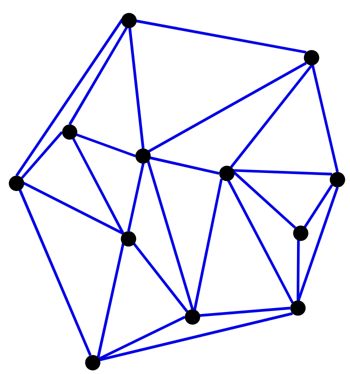
Based on:



Computational Geometry: Algorithms and Applications

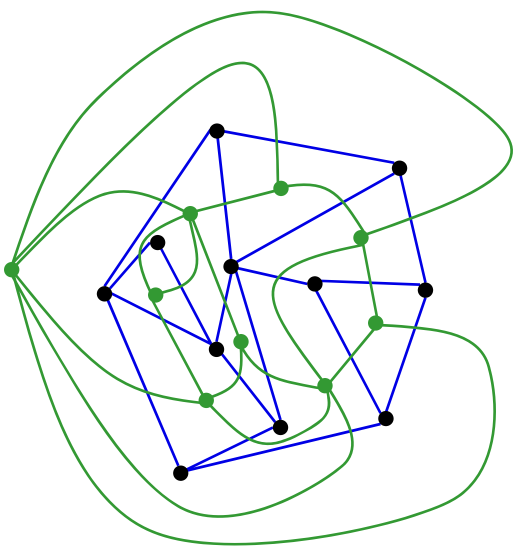
Triangulation

- Let $P = \{p_1, \dots, p_n\} \subseteq R^2$ be a finite set of points in the plane.
- A **triangulation of *P*** is a simple, plane (i.e., planar embedded), connected graph T=(P,E) such that
  - every edge in E is a line segment,
  - the outer face is bounded by edges of CH(P),
  - all inner faces are triangles.



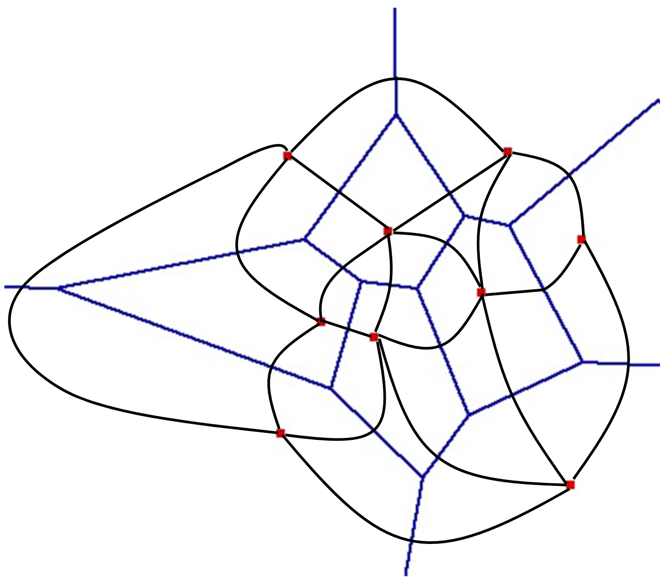
Dual Graph

- Let $G = (V, E)$ be a plane graph. The dual graph G^* has
  - a vertex for every face of G,
  - an edge for every edge of G, between the two faces incident to the original edge

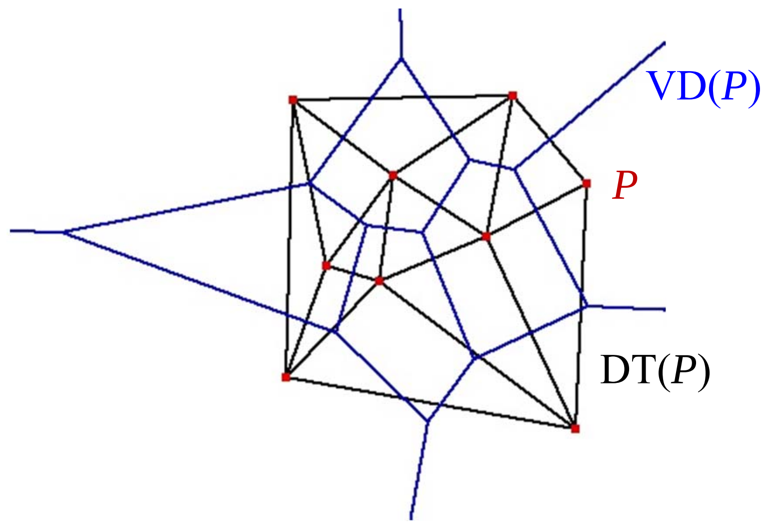


Delaunay Triangulation

- Let *G* be the plane graph for the Voronoi diagram VD(*P*) . Then the dual graph *G** is called the **Delaunay Triangulation DT(*P*)**.



Canonical straight-line embedding for DT(P):

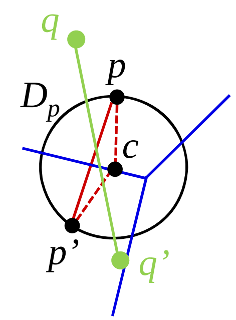


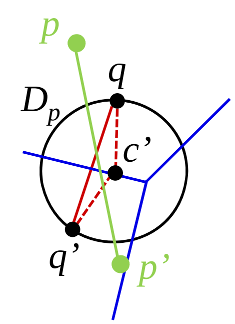
- If *P* is in general position (no three points on a line, no four points on a circle) then every inner face of DT(*P*) is indeed a triangle.
- DT(*P*) can be stored as an abstract graph, without geometric information. (No such obvious storing scheme for VD(*P*).)

Straight-Line Embedding

- **Lemma:** DT(P) is a plane graph, i.e., the straight-line edges do not intersect.
- **Proof:**
  - $\overline{pp'}$ is an edge of DT(P) ⇔ There is an empty closed disk D_p with p and p' on its boundary, and its center c on the bisector.
  - Let $\overline{qq'}$ be another Delaunay edge that intersects $\overline{pp'}$

    ⇒ q and q' lie outside of D_p, therefore $\overline{qq'}$ also intersects $\overline{pc}$ or $\overline{p'c'}$
  - Similarly, $\overline{pp'}$ also intersects $\overline{qc'}$ or $\overline{q'c'}$





⇒ ($\overline{pc}$ or $\overline{p'c'}$) and ($\overline{qc'}$ or $\overline{q'c'}$) intersect

⇒ The edges are not in different Voronoi cells

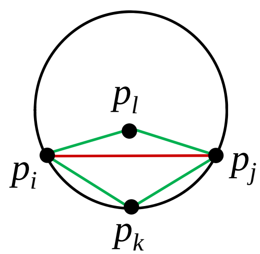
⇒ Contradiction

□

Characterization I of DT(P)

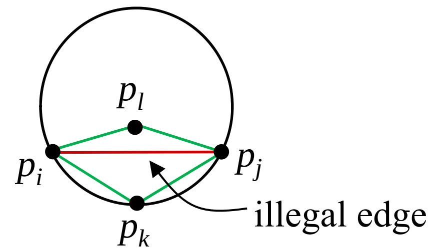
- **Lemma**: Let $p,q,r \in P$ let Δ be the triangle they define. Then the following statements are equivalent:
  a) Δ belongs to DT(P)
  b) The circumcenter of Δ is a vertex in VD(P)
  c) The circumcircle of Δ is empty (i.e., contains no other point of P)

- **Characterization I**: Let T be a triangulation of P.
  Then T=DT(P) $\Leftrightarrow$ The circumcircle of any triangle in T is empty.

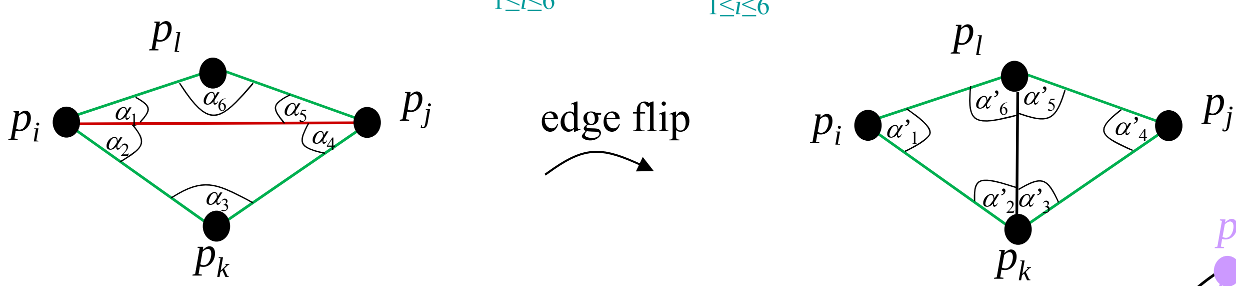


Illegal Edges

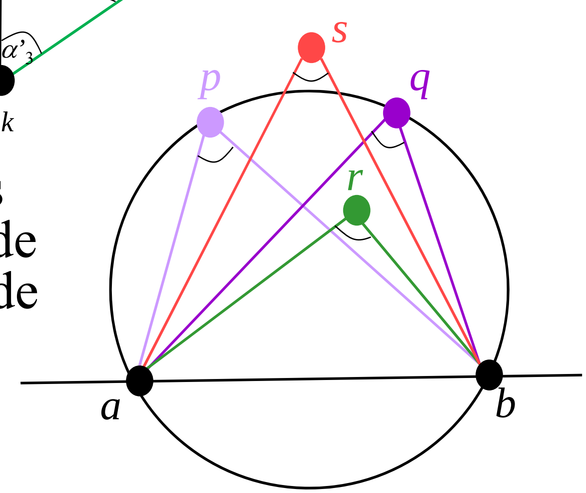
- **Definition:** Let $p_i, p_j, p_k, p_l \in P$.
Then $\overline{p_i p_j}$ is an **illegal edge** $\Leftrightarrow$ p_l lies in the interior of the circle through p_i, p_j, p_k .



- **Lemma:** Let $p_i, p_j, p_k, p_l \in P$.
Then $p_i p_j$ is **illegal** $\Leftrightarrow$ $\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i$



- **Theorem (Thales):** Let a, b, p, q be four points on a circle, and let r be inside and let s be outside of the circle, such that p,q,r,s lie on the same side of the line through a, b.
Then $\angle a,s,b < \angle a,q,b = \angle a,p,b < \angle a,r,b$



Characterization II of DT(P)

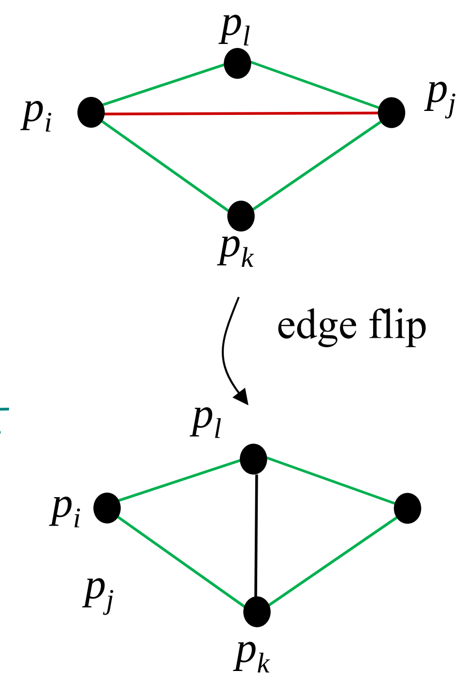
- **Definition:** A triangulation is called legal if it does not contain any illegal edges.
- **Characterization II**: Let T be a triangulation of P. Then T=DT(P) $\Leftrightarrow$ T is legal.
- **Algorithm Legal_Triangulation(T):**

  Input: A triangulation T of a point set P

  Output: A legal triangulation of P

  while T contains an illegal edge $\overline{p_i p_j}$ do
  //Flip $\overline{p_i p_j}$
  Let p_i, p_j, p_k, p_l be the quadrilateral containing $\overline{p_i p_j}$
  Remove $\overline{p_i p_j}$ and add $\overline{p_k p_l}$

  return T



Runtime analysis:

- In every iteration of the loop the angle vector of T (all angles in T sorted by increasing value) increases
- With this one can show that a flipped edge never appears again
- There are O(n^2) edges, therefore the runtime is O(n^2)

Characterization III of DT(P)

- **Definition:** Let T be a triangulation of P and let $\alpha_1, \alpha_2, \ldots, \alpha_{3m}$ be the angles of the m triangles in T sorted by increasing value. Then $A(T)=(\alpha_1,\alpha_2,\ldots,\alpha_{3m})$ is called the angle vector of T.
- **Definition:** A triangulation T is called **angle optimal** $\Leftrightarrow A(T) > A(T')$ for any other triangulation of the same point set P.

- Let T' be a triangulation that contains an illegal edge, and let T'' be the resulting triangulation after flipping this edge. Then $A(T'') > A(T')$.
- T is angle optimal $\Rightarrow$ T is legal $\Rightarrow$ T=DT(P)

- **Characterization III**: Let T be a triangulation of P. Then T=DT(P) $\Leftrightarrow$ T is angle optimal.

  (If P is not in general position, then any triangulation obtained by triangulating the faces maximizes the minimum angle.)