CMPS 2200 – Fall 2012

B-trees

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External memory dictionary

Task: Given a large amount of data that does not fit into main memory, process it into a dictionary data structure

- Need to minimize number of disk accesses
- With each disk read, read a whole block of data
- Construct a balanced search tree that uses one disk block per tree node
- Each node needs to contain more than one key

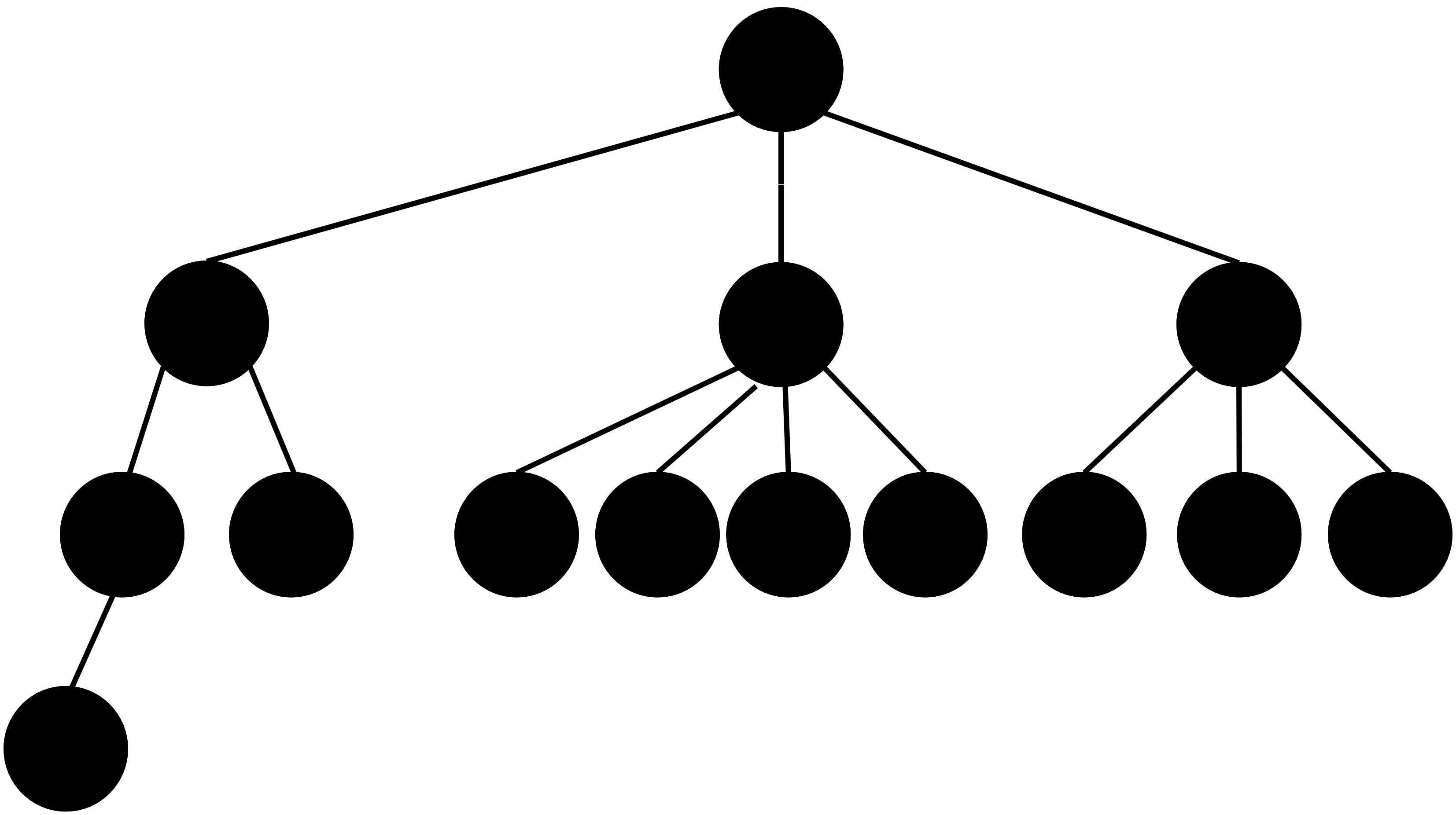
k-ary search trees

A ***k*-ary search tree** T is defined as follows:

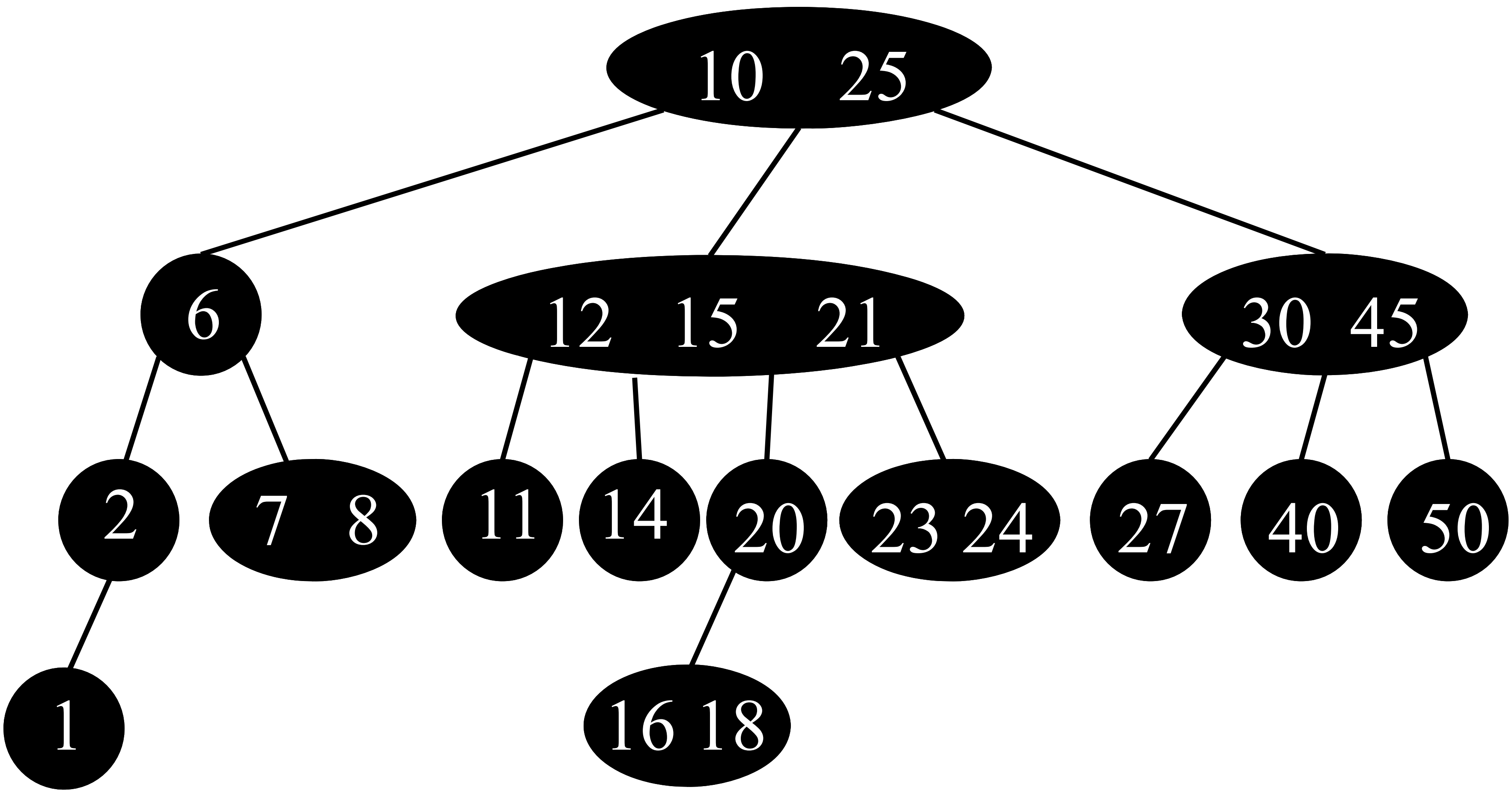
- For each node x of T:
  - x has at most k children (i.e., T is a k-ary tree)
  - x stores an ordered list of pointers to its children, and an ordered list of keys
  - For every internal node: #keys = #children-1
  - x fulfills the search tree property:

  keys in subtree rooted at i-th child $\leq$ i-th key $<$ keys in subtree rooted at $(i+1)$-st child

Example of a 4-ary tree



Example of a 4-ary search tree

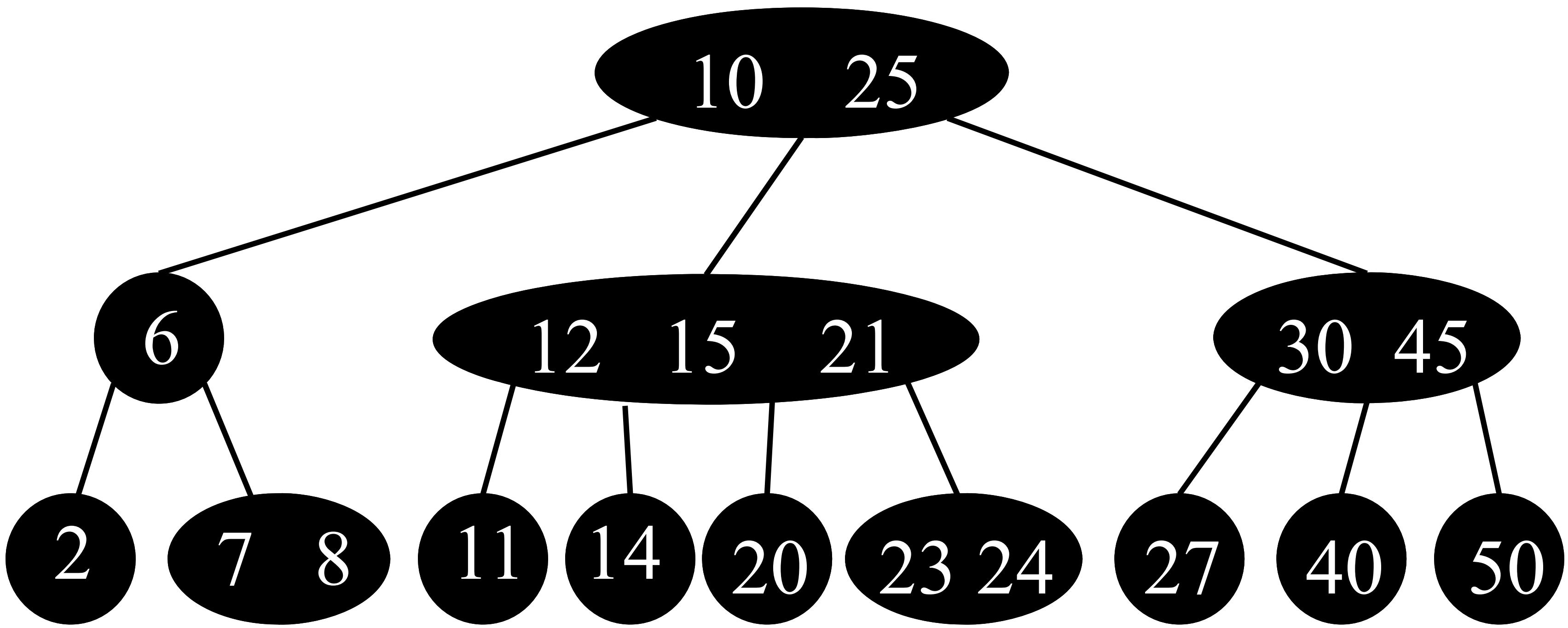


B-tree

A ***B*-tree** T with **minimum degree** $k \geq 2$ is defined as follows:

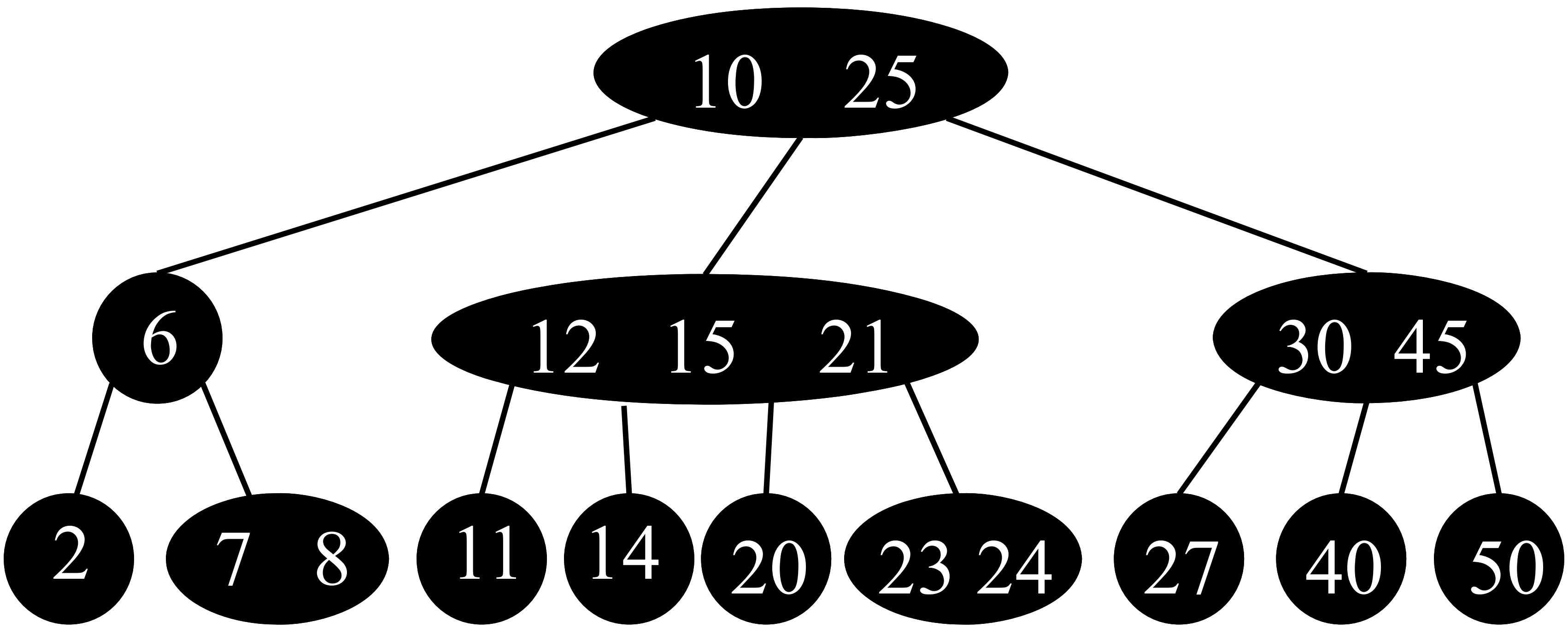
1. T is a $(2k)$-ary search tree
2. Every node, except the root, stores at least k-1 keys
   (every internal non-root node has at least k children)
3. The root must store at least one key
4. All leaves have the same depth

B-tree with *k*=2



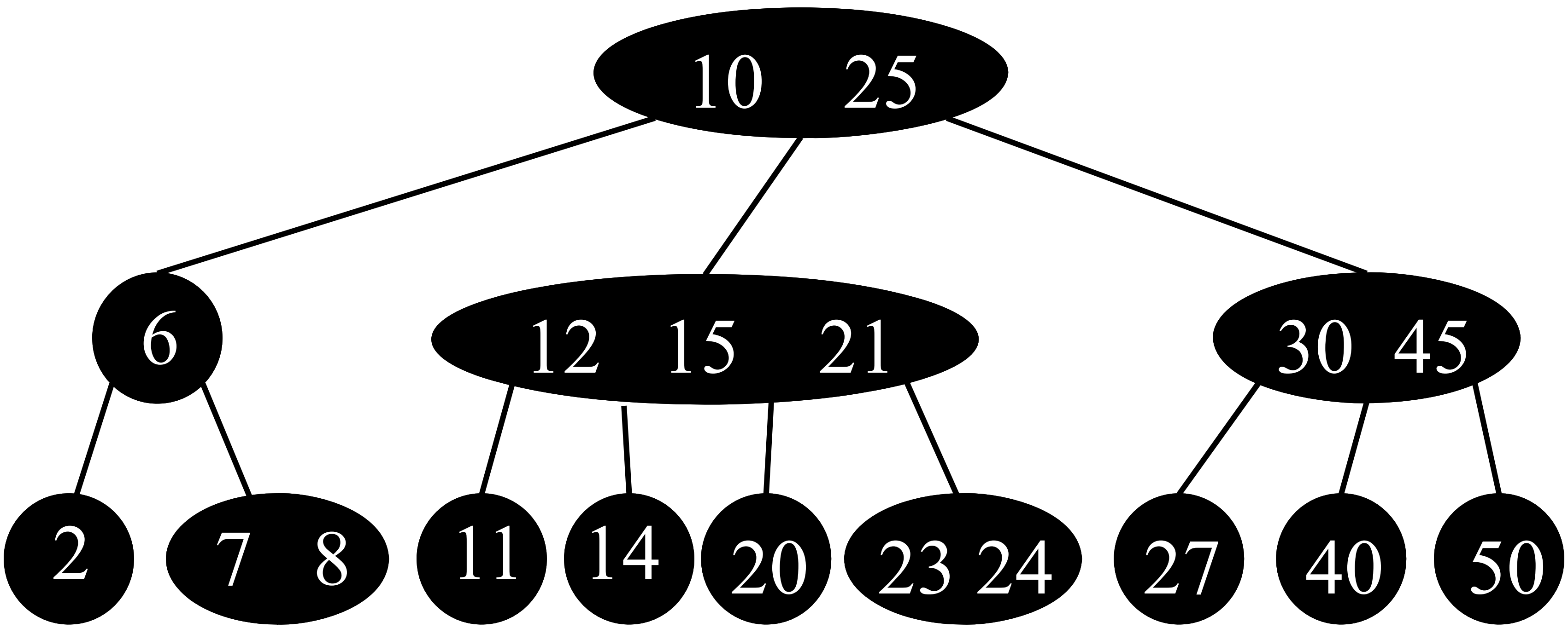
1. T is a $(2k)$-ary search tree

B-tree with *k*=2



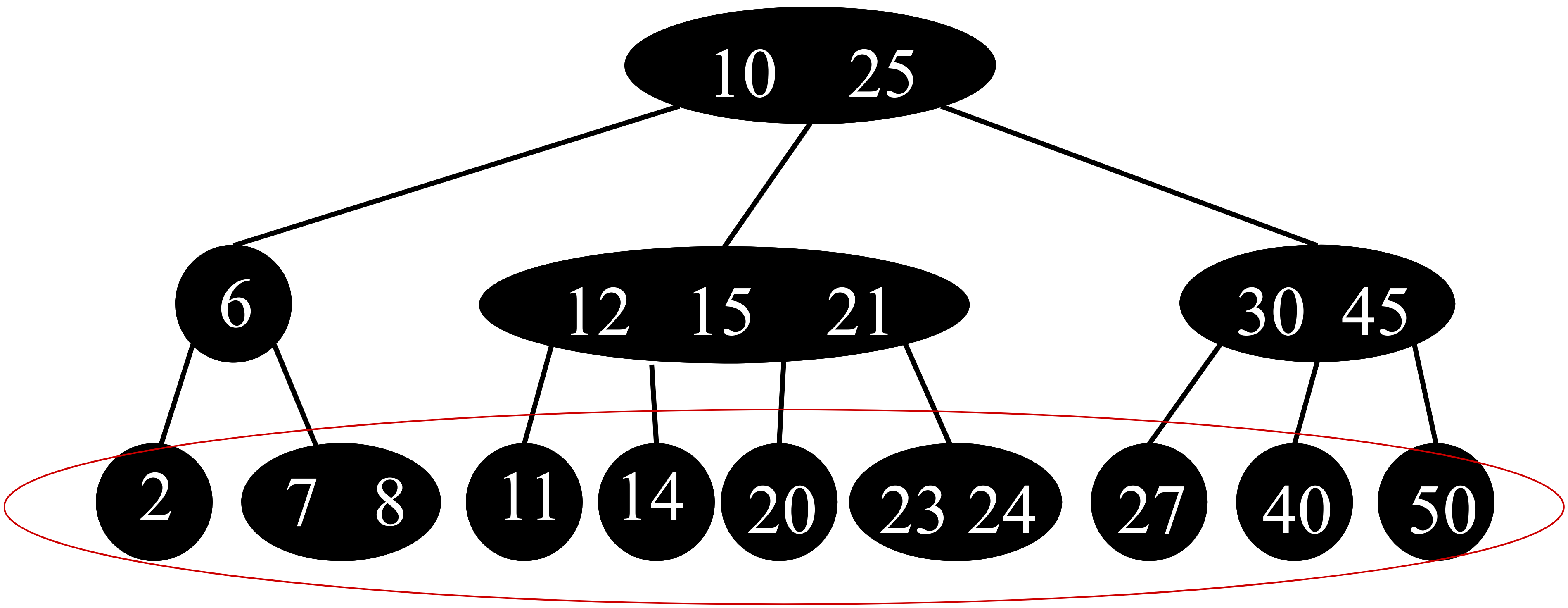
2. Every node, except the root, stores at least *k*-1 keys

B-tree with *k*=2



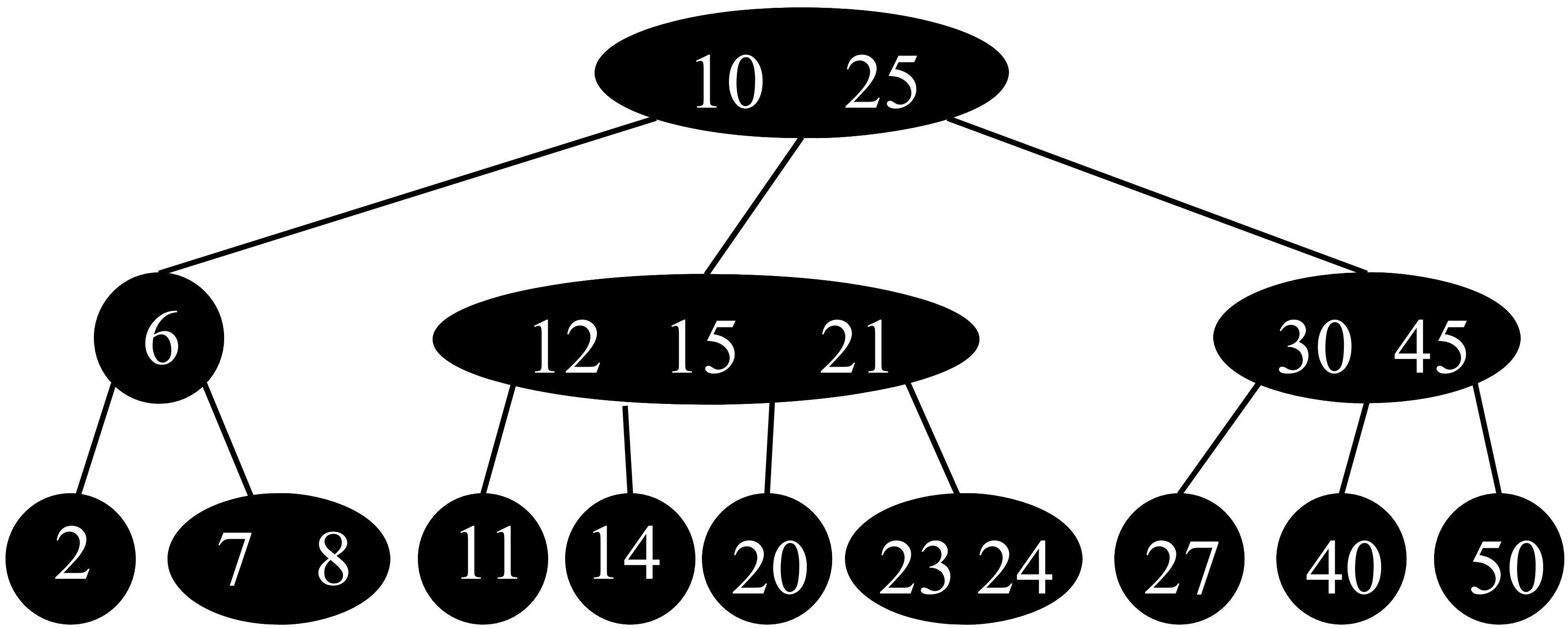
3. The root must store at least one key

B-tree with k=2



4. All leaves have the same depth

B-tree with *k*=2



Remark: This is a (2,3,4)-tree.

Height of a B-tree

Theorem: For a B-tree with minimum degree $k \geq 2$ which stores n keys has height h holds:

$$h \leq \log_k (n+1)/2$$

Proof: #nodes $\geq 1+2+2k+2k^2+\ldots+2k^{h-1}$

level 0 level 1 level 2 level 3

$$n = \#\text{keys} \geq 1+(k-1)\sum_{i=0}^{h-1} 2k^i = 1+2(k-1)\cdot\frac{k^h-1}{k-1} = 2k^h-1$$

B-tree search

B-TREE-SEARCH(x,key)

$i \leftarrow 1$

while $i \leq \#keys\ of\ x$ **and** $key > i$-th key of x

do $i \leftarrow i+1$

if $i \leq \#keys\ of\ x$ **and** $key = i$-th key of x

then return (x,i)

if x is a leaf

then return NIL

else b=DISK-READ(i-th child of x)

return B-TREE-SEARCH(b,key)

B-tree search runtime

- $O(k)$ per node
- Path has height $h = O(\log_k n)$
- CPU-time: $O(k \log_k n)$

- Disk accesses: $O(\log_k n)$

  disk accesses are more expensive than CPU time

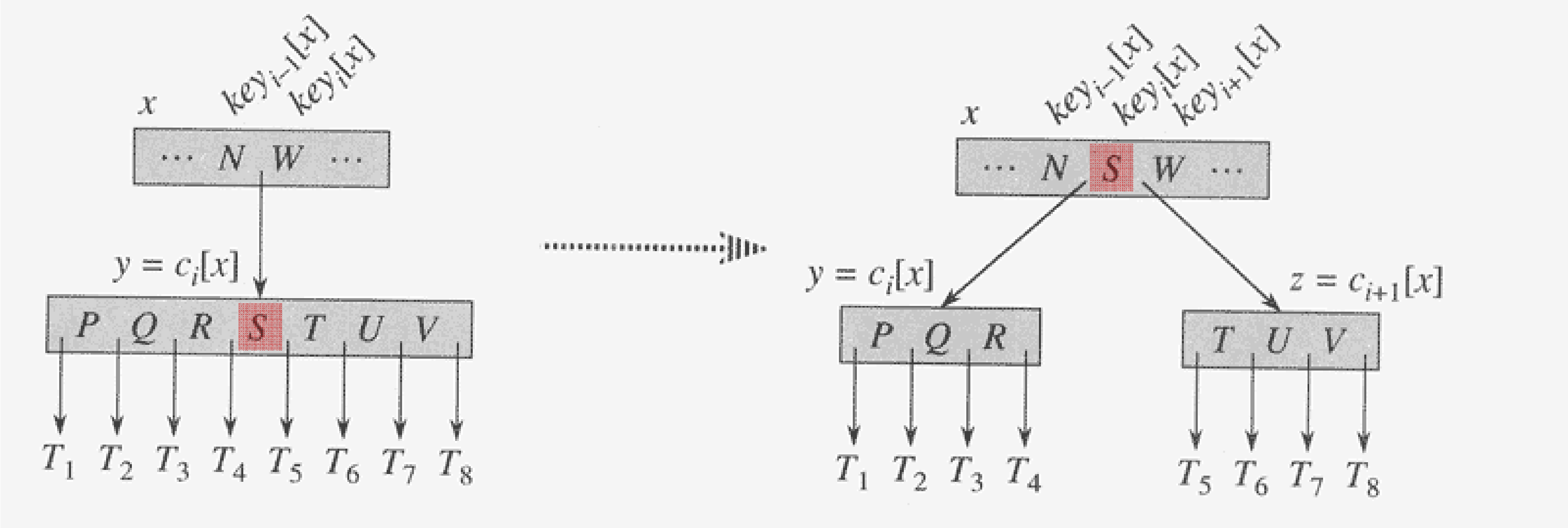
B-tree insert

- There are different insertion strategies. We just cover one of them
- Make one pass down the tree:
  - The goal is to insert the new *key* into a leaf
  - Search where *key* should be inserted
  - **Only descend into non-full nodes:**
    - If a node is full, split it. Then continue descending.
    - **Splitting of the root node is the only way a B-tree grows in height**

B-TREE-SPLIT-CHILD(x,i,y)

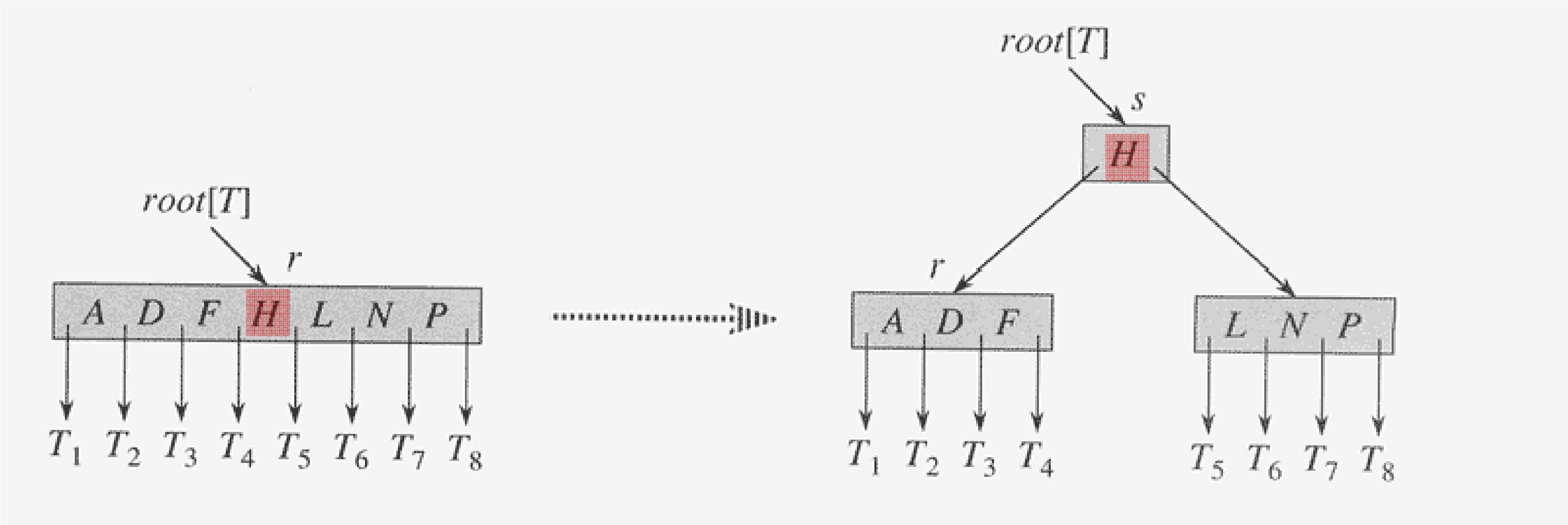
has $2k$-1 keys

- Split full node y into two nodes y and z of k-1 keys
- Median key S of y is moved up into y's parent x
- Example below for k = 4



Split root: B-TREE-SPLIT-CHILD(s,1,r)

- The **full** root node r is split in two.
- A new root node s is created
- s contains the median key H of r and has the two halves of r as children
- Example below for $k = 4$



B-TREE-INSERT(T,key)

r = root[T]

if (# keys in r) = $2k$-1 // root r is full

    //insert new root node:

    s ← ALLOCATE-NODE()

    root[T] ← s

    // split old root r to be two children of new root s

    B-TREE-SPLIT-CHILD(s,1,r)

    B-TREE-INSERT-NONFULL(s,key)

else B-TREE-INSERT-NONFULL(r,key)

B-TREE-INSERT-NONFULL(x,key)

if x is a leaf **then**

    insert key at the correct (sorted) position in x

    DISK-WRITE(x)

else

    find child c of x which by the search tree property should contain key

    DISK-READ(c)

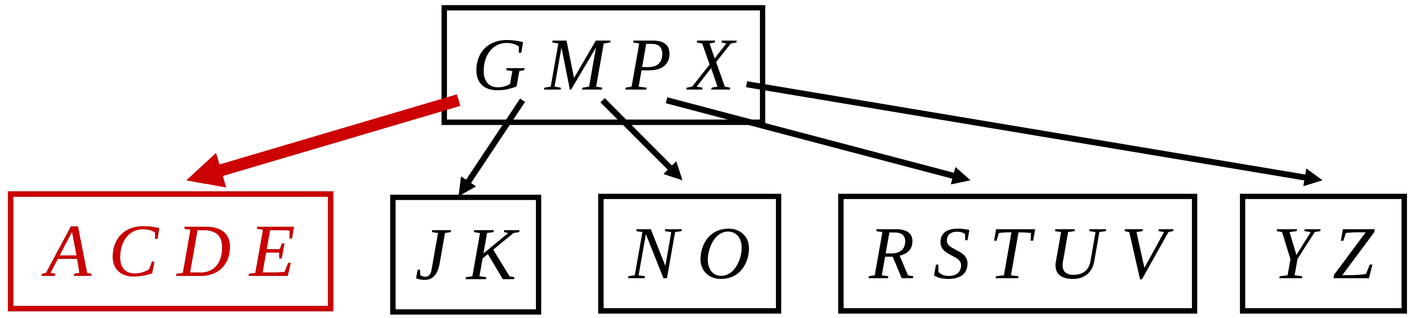
    if c is full **then** // c contains $2k$-1 keys

        B-TREE-SPLIT-CHILD(x,i,c)

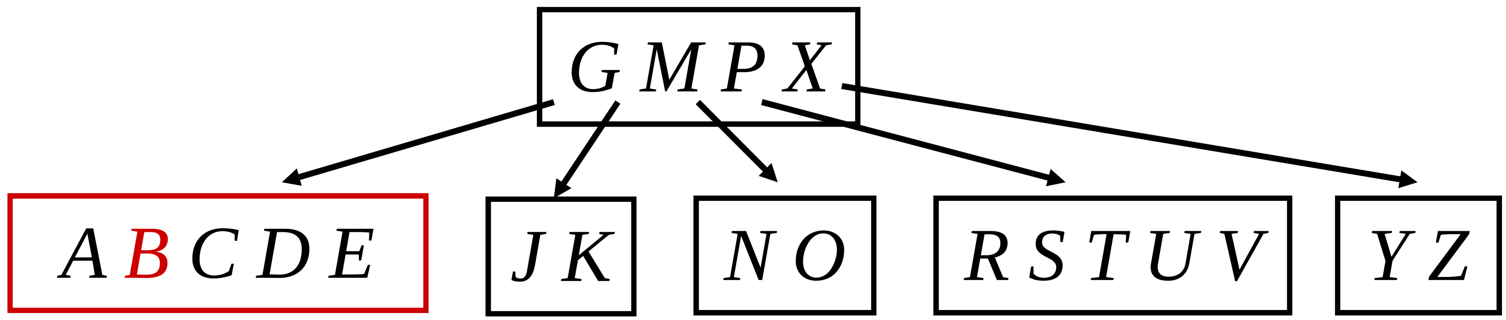
        c=child of x which should contain key

    B-TREE-INSERT-NONFULL(c,key)

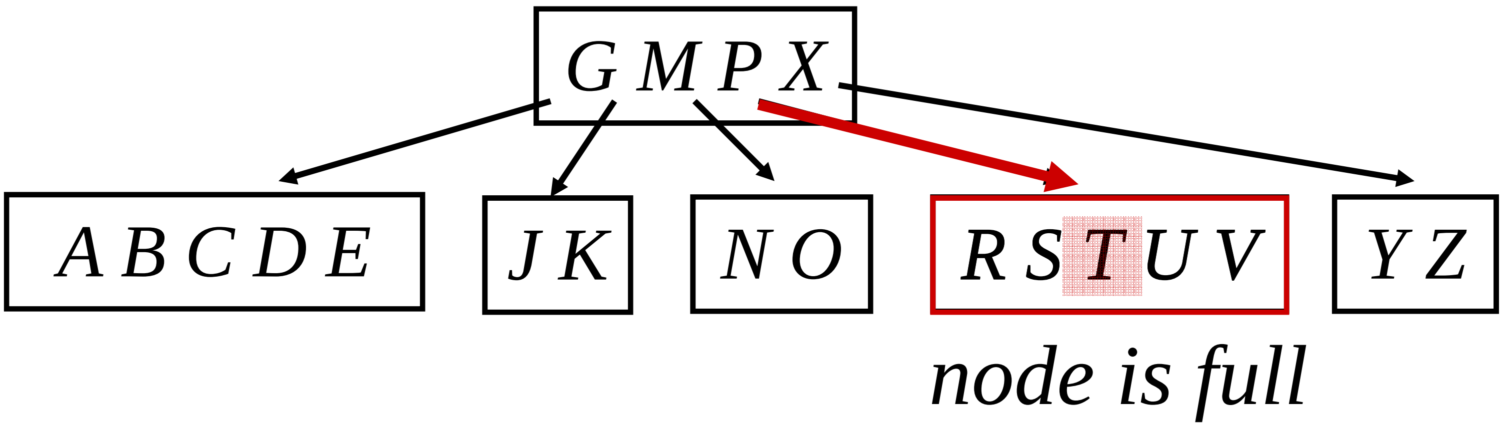
Insert example (k=3)



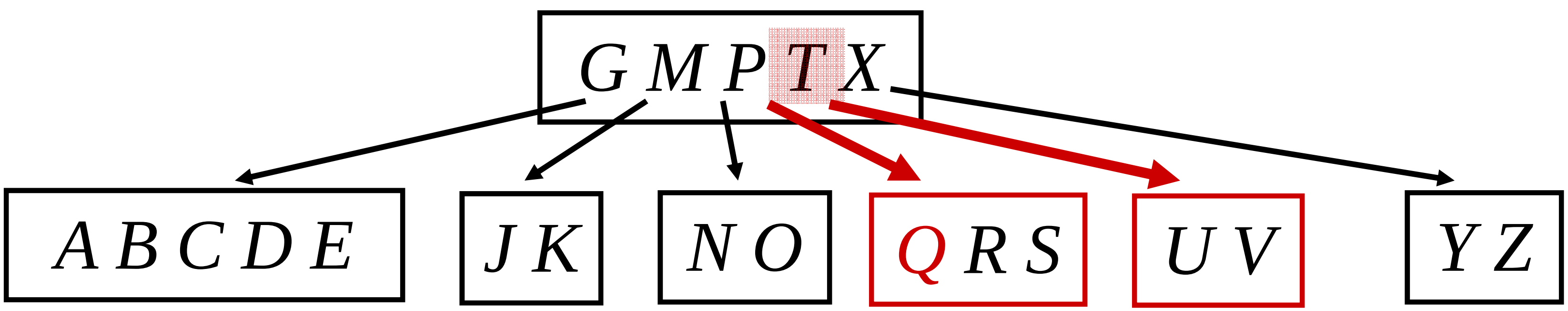
- Insert B:



Insert example (k=3) -- cont.

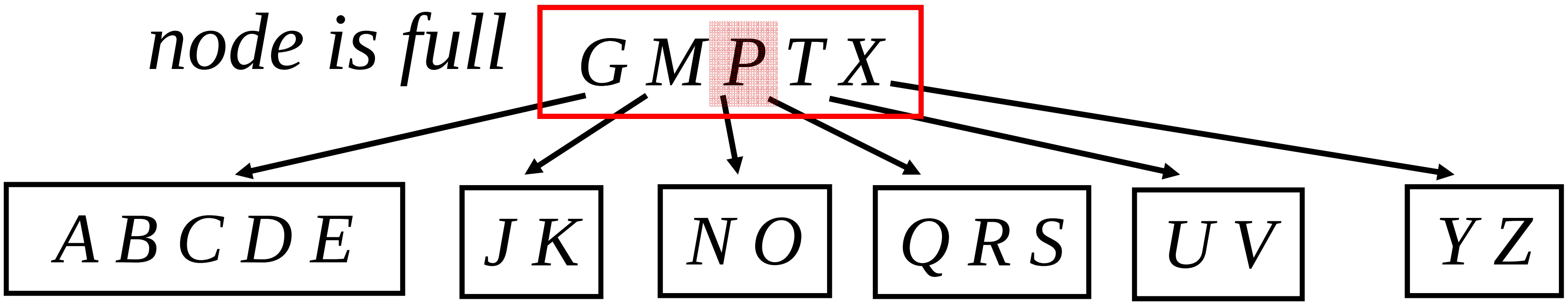


- Insert *Q*:

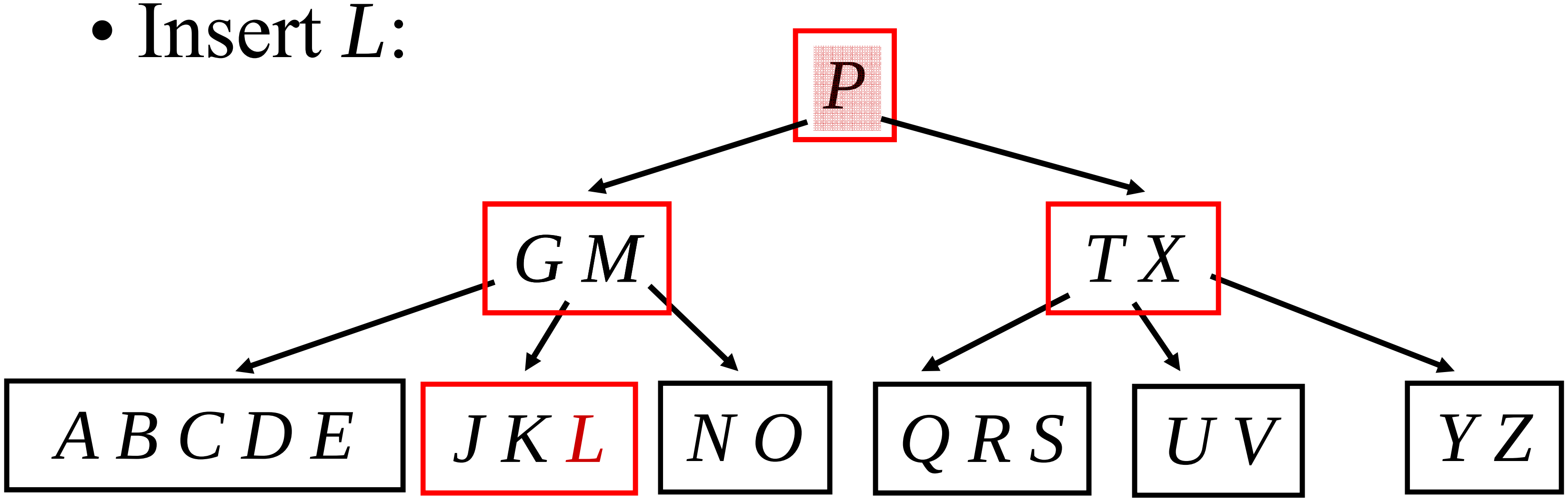


Insert example (k=3) -- cont.

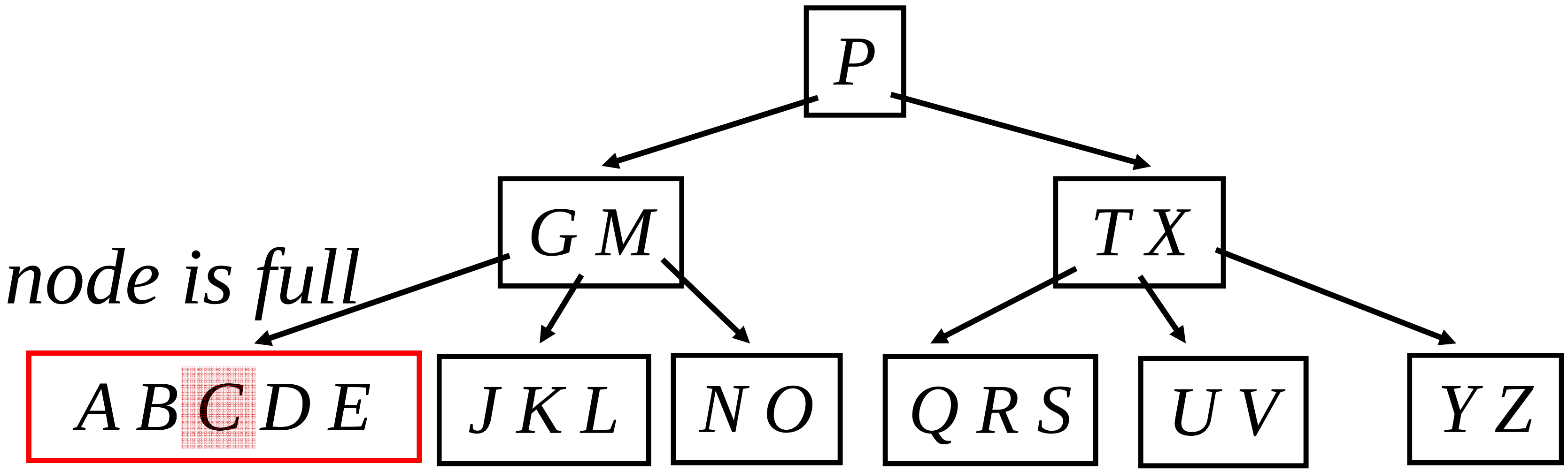
node is full



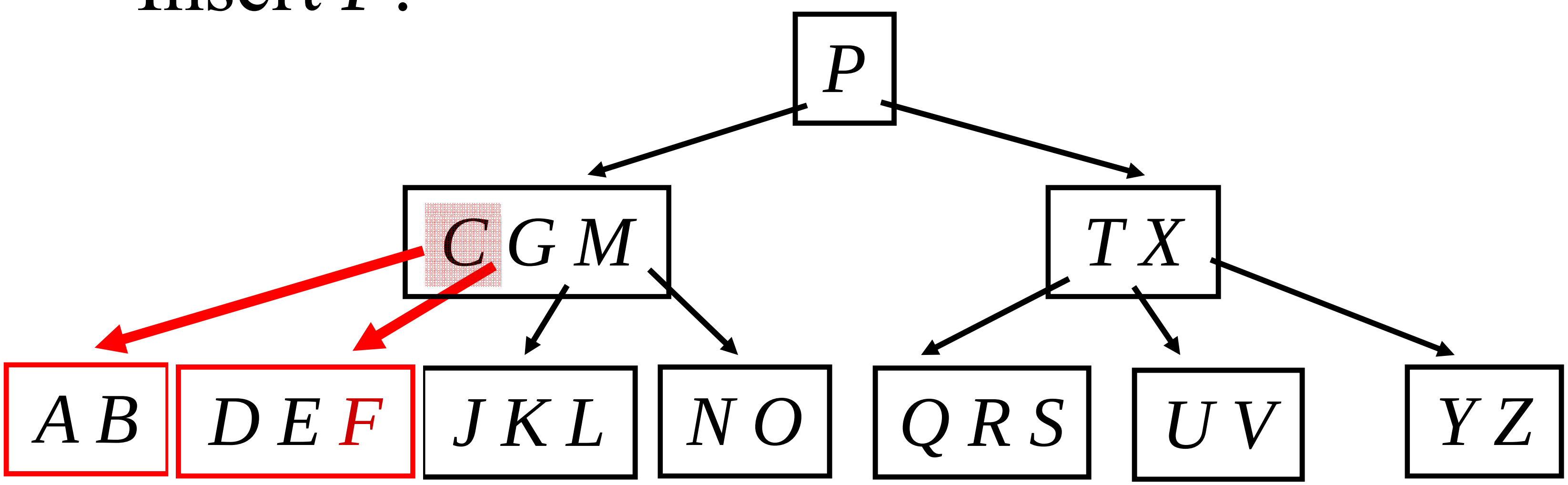
- Insert *L*:



Insert example (k=3) -- cont.



- Insert *F*:



Runtime of B-TREE-INSERT

- $O(k)$ runtime per node
- Path has height $h = O(\log_k n)$
- CPU-time: $O(k \log_k n)$

- Disk accesses: $O(\log_k n)$

  disk accesses are more expensive than CPU time

Deletion of an element

- Similar to insertion, but a bit more complicated
- If sibling nodes get not full enough, they are **merged** into a single node
- Same complexity as insertion

B-trees -- Conclusion

- B-trees are balanced $2k$-ary search trees
- The **degree** of each node is **bounded from above and below** using the parameter k
- All leaves are at the same height
- **No rotations** are needed: During insertion (or deletion) the balance is maintained by node **splitting** (or node **merging**)
- The tree grows (shrinks) in height only by splitting (or merging) the root