9/24/12

3. Homework

Due **10/2/12** in the lab

1. **Master theorem (8 points)**
   Use the master theorem to solve the following recurrences. Justify your results. Assume that $T(1) = 1$.

   (a) $T(n) = 16T(\frac{n}{2}) + n^2$
   (b) $T(n) = T(\frac{n}{2}) + \sqrt{n}$
   (c) $T(n) = 16T(\frac{n}{4}) + n^2 \log n$
   (d) $T(n) = 81T(\frac{n}{3}) + n^2 \log n$

2. **Randomized insertion sort (8 points)**
   The goal of this problem is to compute the expected runtime of randomized insertion sort.

   (a) (4 points) For any $i = 1, \ldots, n$ let $X_i : S \to \mathbb{R}$ be the runtime random variable for the i-th iteration of the algorithm. Compute $E(X_i)$. *(Hint: Use the second definition of expected value. This is not just a simple indicator random variable.)*
   (b) (4 points) Let $X : S \to \mathbb{R}$ be the runtime random variable for the whole algorithm. Use linearity of expectation to compute the expected runtime $E(X)$.

3. **Random permutation (8 points)**

   (a) (3 points) Give pseudocode to compute a random permutation of an array of n distinct numbers. What is the runtime of your algorithm? (Can you make the algorithm be in-place, i.e., using only additional constant space?)
   (b) (5 points) Assume you are given an array $A[1..n]$ of n distinct numbers, and you compute a random permutation of it. Use linearity of expectation to compute the expected number of fixpoints (indices that contain the same number before and after).

   Clearly describe the sample space and the random variable that you are using, and break your overall random variable into multiple (indicator) random variables.

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Practice Problems

(Not required for homework credit.)

1. **Master theorem**
   Use the master theorem to solve the following recurrences. Justify your results. Assume that $T(1) = 1$.

   (a) $T(n) = 125T(\frac{n}{5}) + 1$
   (b) $T(n) = 9T(\frac{n}{3}) + n^4$
   (c) $T(n) = 2T(\frac{n}{4}) + \sqrt{n}\log n$

2. **Expected values of dice**
   Clearly describe the sample space and the random variables you use.

   (a) Compute the expected value of rolling a fair four-sided die.
   (b) Compute the expected value of the sum of two fair four-sided dies...
      i. ... using the definition of the expected value.
      ii. ... using linearity of expectation. *(Hint: Express your random variable as the sum of two random variables.)*
   (c) Use linearity of expectation to compute the expected value of the sum of k fair four-sided dice, for any $k \geq 1$. *(Hint: Express your random variable as the sum of k random variables.)*

3. SimpleRoulette

   The game SimpleRoulette is played as follows: The roulette wheel has a slot for each number from 1 to 36 as well as a slot for 0 and for 00. You can bet on any number between 1 and 36, but not on 0 or 00. A bet costs you \$10. If the ball drops on the slot with your number, you get paid \$360, otherwise you don't get paid anything.

   Assuming that the wheel is fair (i.e., all numbers are equally likely), what is your expected win/loss in this game?