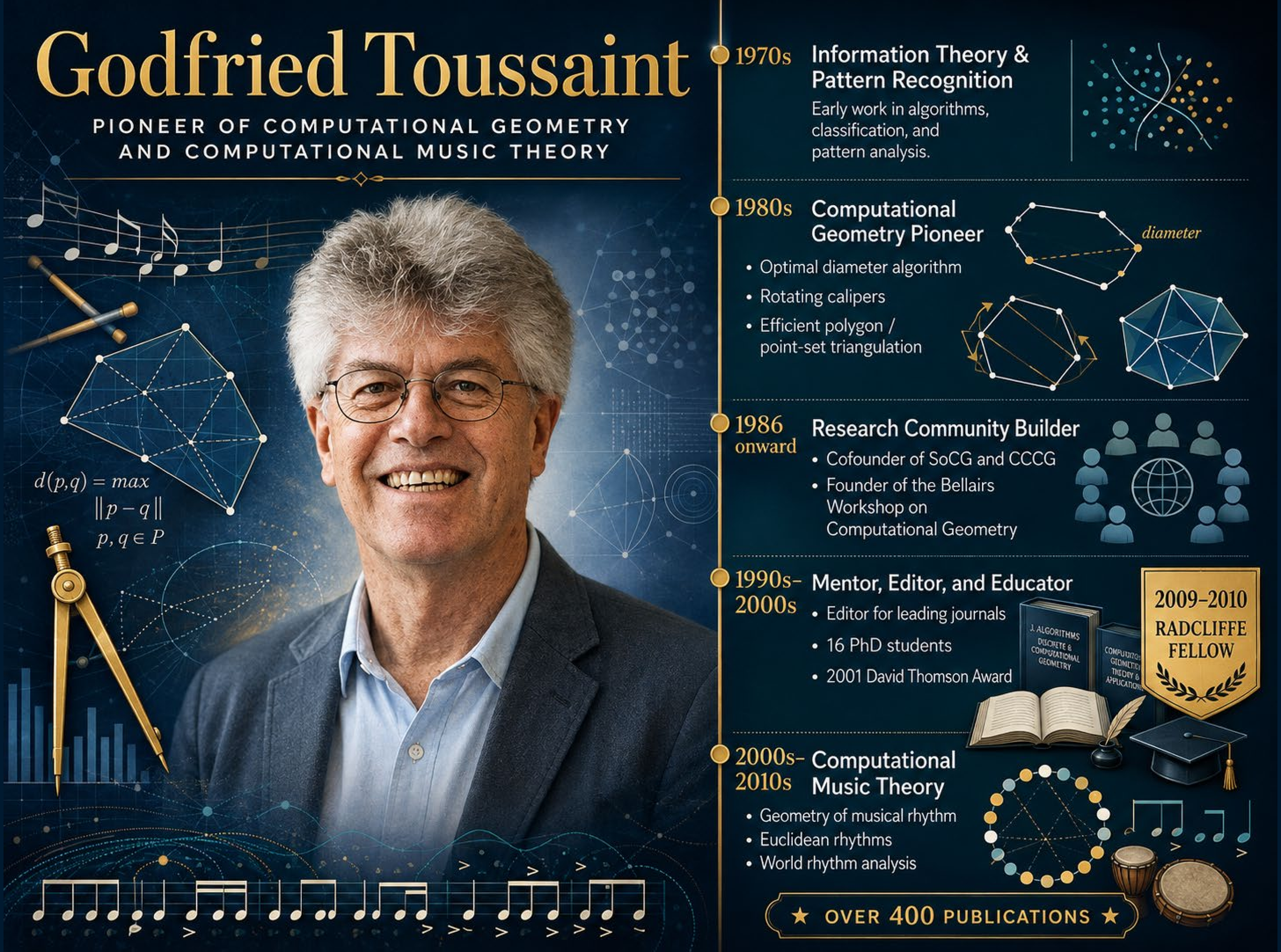
Geometric Distances for Curves and Graphs: From Matching to Simplification

Carola Wenk
Tulane University

With thanks to many collaborators and the National Science Foundation for supporting this research.


Godfried Toussaint
PIONEER OF COMPUTATIONAL GEOMETRY AND COMPUTATIONAL MUSIC THEORY
$d(p,q) = \max_{p,q \in P} \|p - q\|$
1970s
Information Theory & Pattern Recognition
Early work in algorithms, classification, and pattern analysis.
1980s
Computational Geometry Pioneer
• Optimal diameter algorithm
• Rotating calipers
• Efficient polygon / point-set triangulation
diameter
1986 onward
Research Community Builder
• Cofounder of SoCG and CCCG
• Founder of the Bellairs Workshop on Computational Geometry
1990s–2000s
Mentor, Editor, and Educator
• Editor for leading journals
• 16 PhD students
• 2001 David Thomson Award
2009–2010 RADCLIFFE FELLOW
2000s–2010s
Computational Music Theory
• Geometry of musical rhythm
• Euclidean rhythms
• World rhythm analysis
★ OVER 400 PUBLICATIONS ★


Carola Wenk
ALGORITHMS, COMPUTATIONAL GEOMETRY, AND INTERPRETABLE SPATIAL DATA ANALYSIS
SHAPE MATCHING
Fréchet Distance
$d_F(P,Q) = \inf_{\alpha,\beta} \max_t \| P(\alpha(t)) - Q(\beta(t)) \|$
MAP MATCHING ROAD NETWORKS
TRAJECTORY ANALYSIS
TOPOLOGICAL DATA ANALYSIS
death
birth
Persistence Diagram
Shape • Holes • Connectivity
GEOMETRIC GRAPHS
ROAD-NETWORK CONSTRUCTION
DIGITAL PATHOLOGY & BIOMEDICAL IMAGING
MOBILITY DATA
1990s–2002
Foundations in Geometry
• Mathematics studies at Freie Universität Berlin
• Ph.D. in Computer Science, 2002
• Dissertation: "Shape Matching in Higher Dimensions"
FREIE UNIVERSITÄT BERLIN
2002–2012
Early Career and Shape Matching
• University of Arizona and UTSA
• Geometric algorithms and Fréchet distance
• Shape matching and similarity of curves
• 2007 NSF CAREER Award
UTSA
2012–2020
Tulane and Interdisciplinary Impact
• Joined Tulane University
• Map matching and road-network construction
• Geospatial data and transportation systems
• Computational biology and medical applications
TULANE UNIVERSITY
2021–2026
Leadership and New Directions
• Chair, Department of Computer Science
• Geometric and topological shape comparison
• Trajectory analysis and geometric graphs
• NIH, NSF, DARPA, and IARPA projects
2023–2025
Recent Recognition
• 2023 ACM SIGSPATIAL Best Data Paper
• 2024 IEEE VIS Best Paper
• 2024 ACM SIGSPATIAL 10-Year Impact Award
• 2025 CCCG Best Paper
Map Construction Algorithms
Map Construction Algorithms (2015)
RESEARCH THEMES: FRÉCHET DISTANCE • MAP MATCHING • TRAJECTORY ANALYSIS • TOPOLOGICAL DATA ANALYSIS • BIOMEDICAL IMAGING

Overview

1. **Shapes Introduction**
   Geometric shapes (curves, graphs), distances, matching, simplification

2. **Distance Measures for Geometric Curves and Graphs**
   Hausdorff, Fréchet, graph distances, …

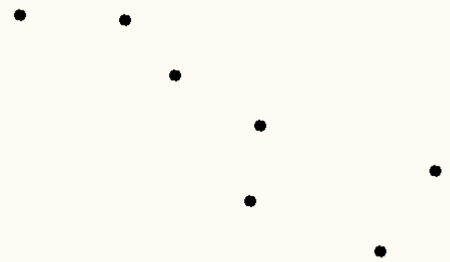
3. **Shape Matching under Transformations**
   Configuration space approaches, approximations, …

4. **Shape Simplification**
   Local vs global simplification, …

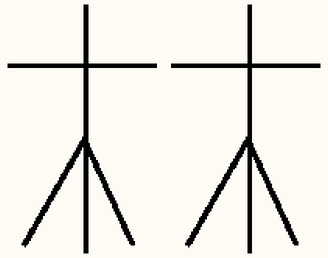
1. **Shapes Introduction**
   Geometric shapes: points, line segments, curves, geometric graphs, …

Discrete Geometric Shapes, Curves, Graphs

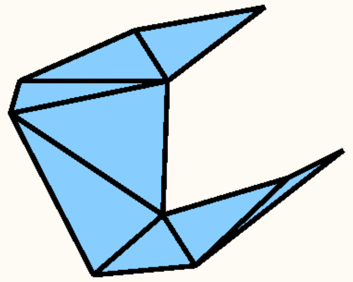
Discrete Geometric Shapes



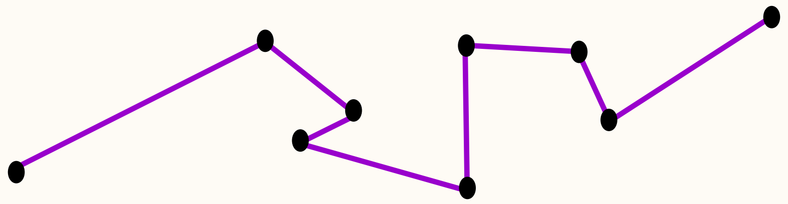
points

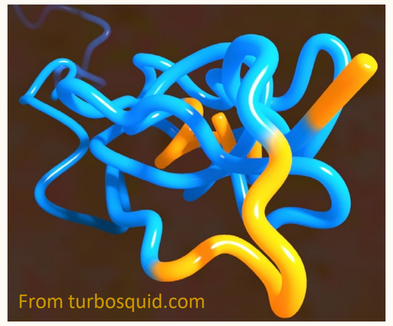


line segments

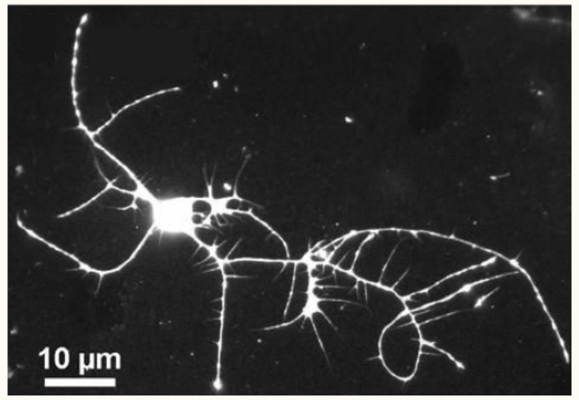


triangles

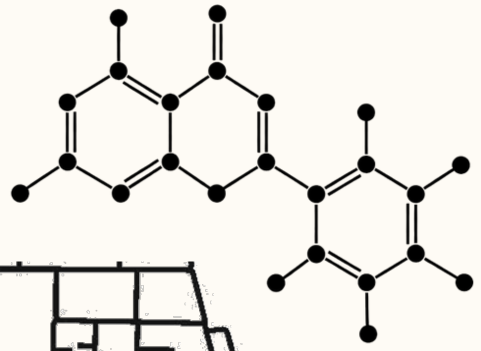


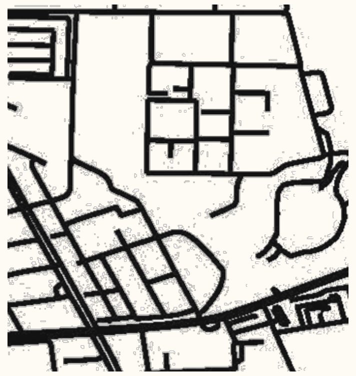


curves (GPS trajectories, protein chains)



geometric trees (plant morphology, neuron)

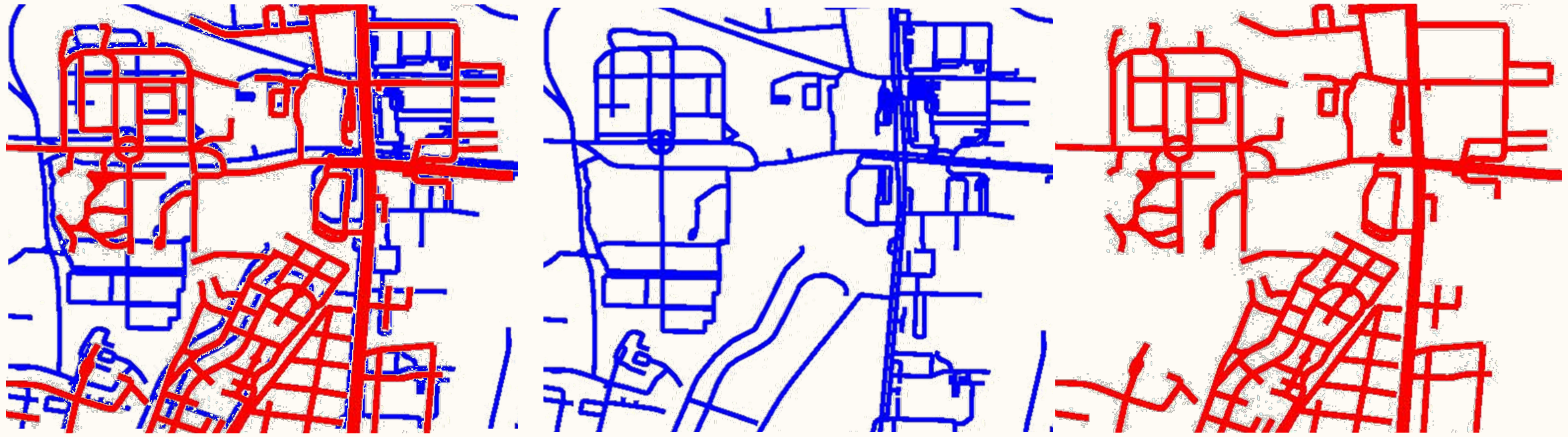




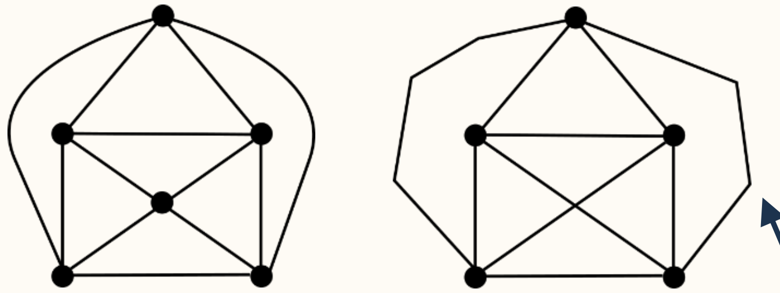
geometric graphs

1D Continuous Embedded Data

- There are lots of distance measures and algorithms for curves but not so many for graphs.
- Graphs are the most general 1D shape.



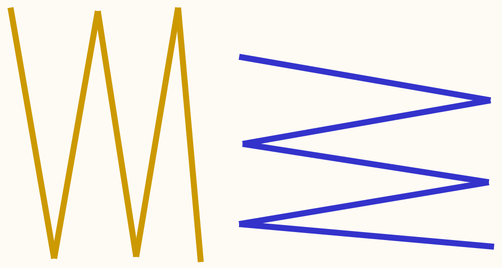
- **Curve:** $f: [0,1] \rightarrow \mathbb{R}^D$.
- **Geometric graph** $G = (V_G, E_G, \Phi_G)$ with continuous $\Phi_G: \mathrm{G} \rightarrow \mathbb{R}^{\mathrm{D}}$ that is homeomorphic (embedded graph) or locally homeomorphic (immersed graph) onto its image.



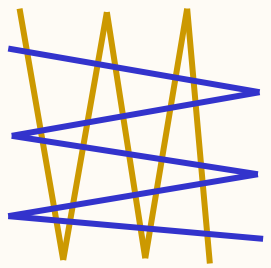
- Vertices mapped to points
- Edges mapped to curves
- May ignore degree-2 vertices

Shape Matching & Simplification

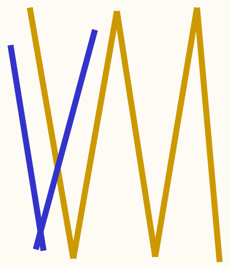
How Similar are Two Shapes?



Transformations (translations, rotations)



Choice of distance measure



Full or partial matching

Shape Matching

Let $\mathcal{T}$ be a set (or group) of transformations. E.g.:

- Translations
- rigid motions (= translations & rotations)
- isometries (= rigid motions and reflections)

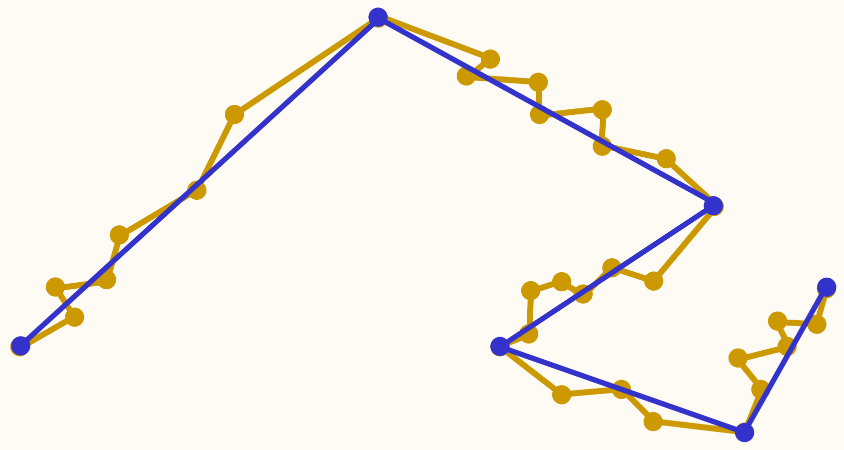
Let $\mathcal{S}$ be a set of shapes.
Let d be a metric or distance on $\mathcal{S}$.

Then given two shapes $\mathrm{A}, \mathrm{B} \in \mathcal{S}$, the **shape matching problem** is to find an optimal transformation $\mathrm{T}^* \in \mathcal{T}$ such that:

$$d(T^*(A), B) = \min_{T \in \mathcal{T}} d(T(A), B)$$

Shape Simplification

Given a shape $\mathrm{A} \in \mathcal{S}$, find a shape $\mathrm{B} \in \mathcal{S}$ such that both $|B| = k$ and $d(A, B) = \Delta$ are small.



- Shortcuts between pairs of vertices.

2. **Distance Measures for Geometric Curves and Graphs**
   Hausdorff, Fréchet, graph distances, …

Hausdorff and Fréchet Distances

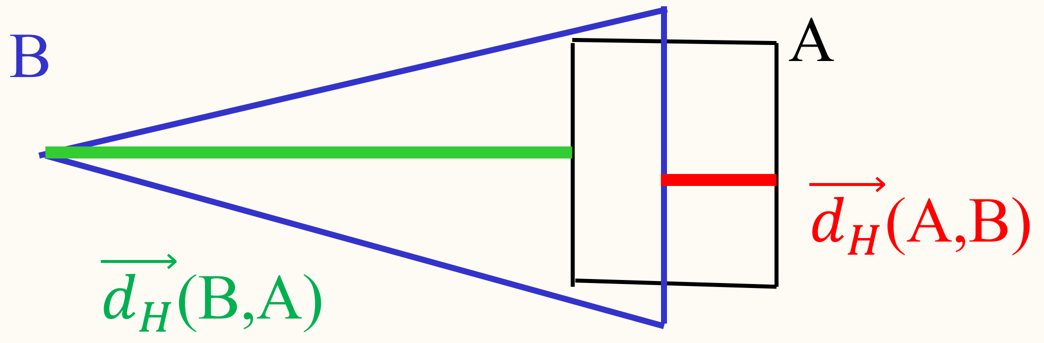
For compact sets A, B and a point metric d:

- The directed Hausdorff distance is:

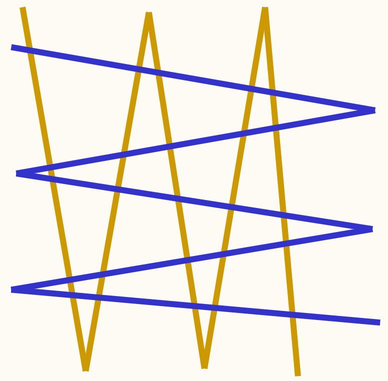
$$\overrightarrow{d_H}(A,B) = \max_{a\in A}\min_{b\in B} d(a,b)$$

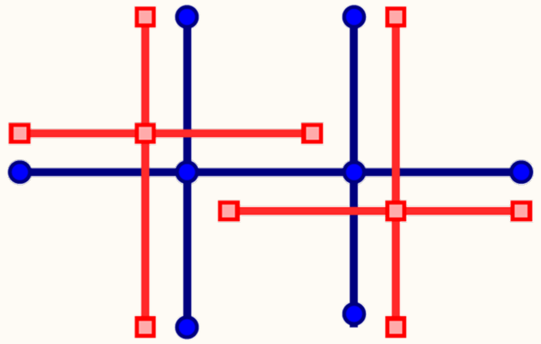
- The undirected Hausdorff distance (metric) is:

$$\mathrm{d}_H(A,B) = \max(\overrightarrow{d_H}(A,B), \overleftarrow{d_H}(A,B))$$



- Point sets in $\mathbb{R}^2$: $O(N \log N)$ via NN search in VDs
- Sets of line segments in $\mathbb{R}^2$: $O(N \log N)$ [ABB95]
- **Con:** Doesn't consider continuity or topology

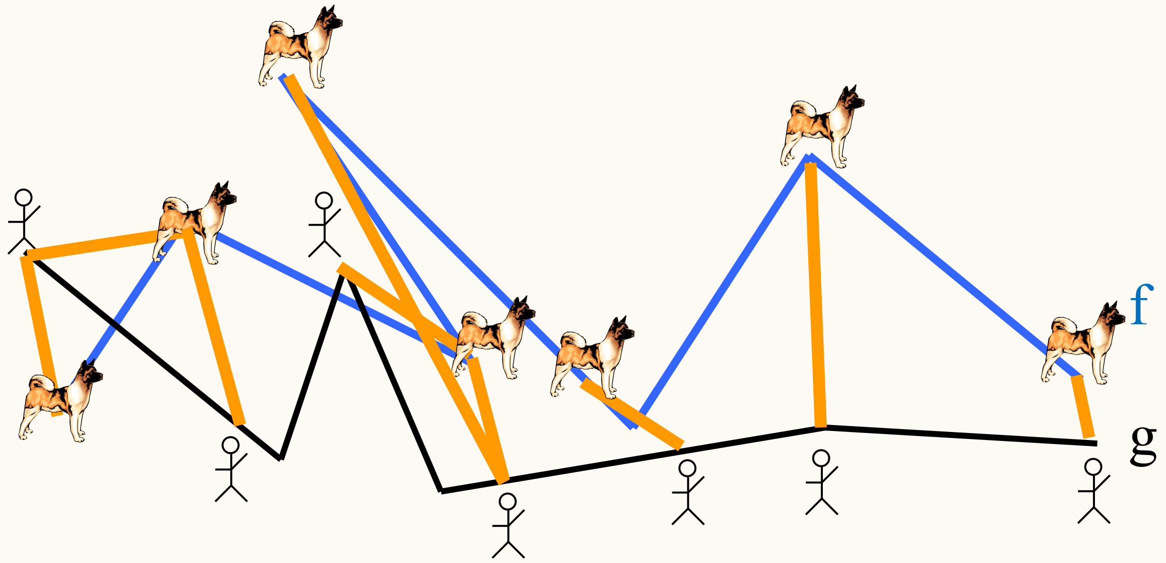




Let $f, g: [0,1] \to \mathbb{R}^D$ be two **curves**. The Fréchet distance is:

$$\mathrm{d}_F(f,g) = \inf_{\alpha,\beta:[0,1]\mapsto[0,1]} \max_{t\in[0,1]} d(f(\alpha(t)), g(\beta(t))),$$

where α, β range over continuous monotone increasing reparameterizations only. (Generalizes to oriented manifolds using orientation-preserving homeomorphisms).

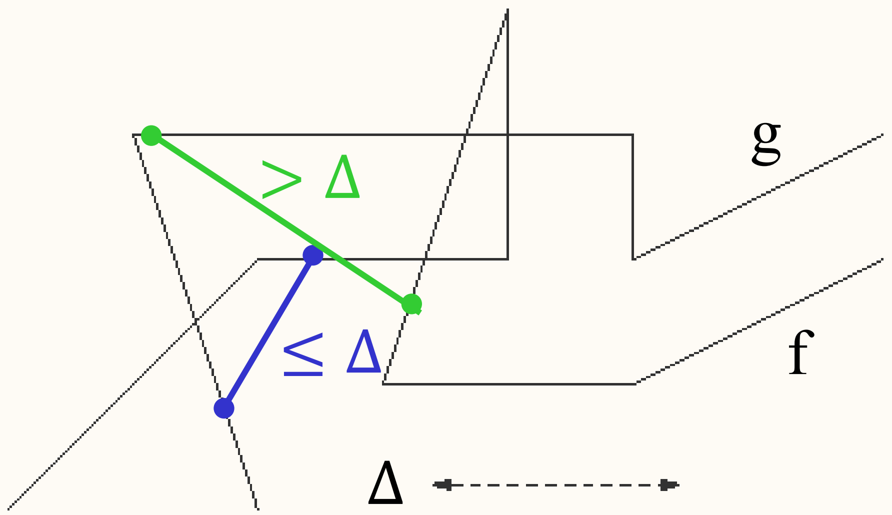


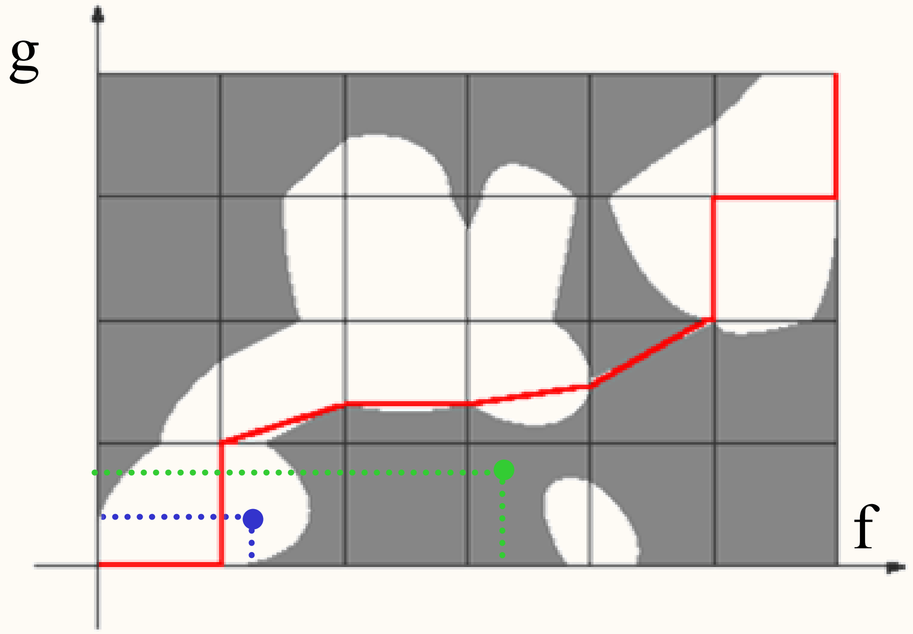
- Man and dog walk on one curve each
- They hold each other at a **leash**
- They are only allowed to go forward
- d_F is the minimal possible leash length

[ABB95] H. Alt, B. Behrends and J. Blömer. Approximate matching of polygonal shapes. Ann. Math. Artif. Intell. 13:251–266, 1995.
[F06] M. Fréchet, Sur quelques points de calcul fonctionel, Rendiconti del Circolo Mathematico di Palermo 22: 1-74, 1906.

Computing the Fréchet Distance

Free space





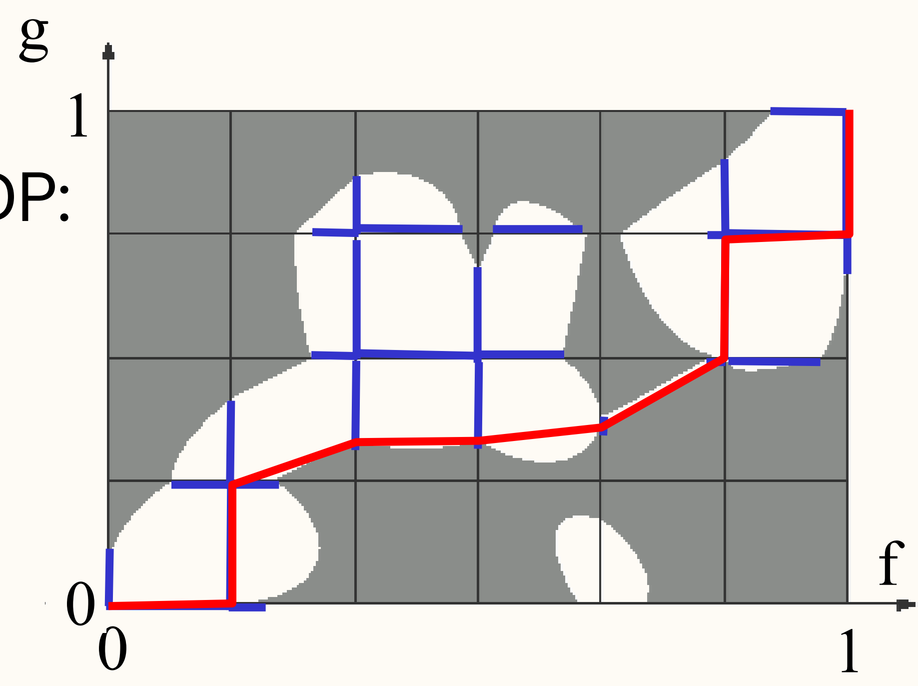
- Let $\Delta > 0$ (eventually solve decision problem)
- $F_\Delta(f,g) = \{(s,t) \in [0,1]^2 : d\big(f(s), g(t)\big) \leq \Delta\}$ is the free space of f and g (white points)
- The free space in one cell is an ellipse

- Monotone path encodes reparameterizations of f, g
- $d_F(f,g) \leq \Delta$ *iff there is a monotone path in the free space from* $(0,0)$ *to* $(1,1)$.

Solve the decision problem

$d_F(f,g) \leq \Delta$ in $O(mn)$ time:

- Find monotone path using DP:
- On each cell boundary compute interval of points reachable by a monotone path from $(0,0)$
- Compute a monotone path by backtracking



Solve the optimization problem

- In practice in O(mn log b) time with binary search and b-bit precision
- In O(mn log mn) time [AG95] using parametric search (using Cole's sorting trick)

Quadratic barrier

- Slightly subquadratic algo: $O(N^2/polylog(N))$ rand. expected [CH25]
- $\Omega(N^{2-\varepsilon})$ cond. lower bound [B14]

[AG95] H. Alt, M. Godau, Computing the Fréchet distance between two polygonal curves, IJCGA 5: 75-91, 1995
[B14] K. Bringmann. Why Walking the Dog Takes Time: Fréchet Distance Has No Strongly Subquadratic Algorithms Unless SETH Fails, FOCS: 661–670, 2014
[CH25] S.-W. Cheng, H. Huang. Fréchet Distance in Subquadratic Time. SODA: 5100–5113, 2025
[S19] Peter Schäfer, Fréchet View – A Tool for Exploring Fréchet Distance Algorithms (Multimedia Exposition), SoCG: 66:1-66:5, 2019.

Fréchet-Like Distances for Graphs

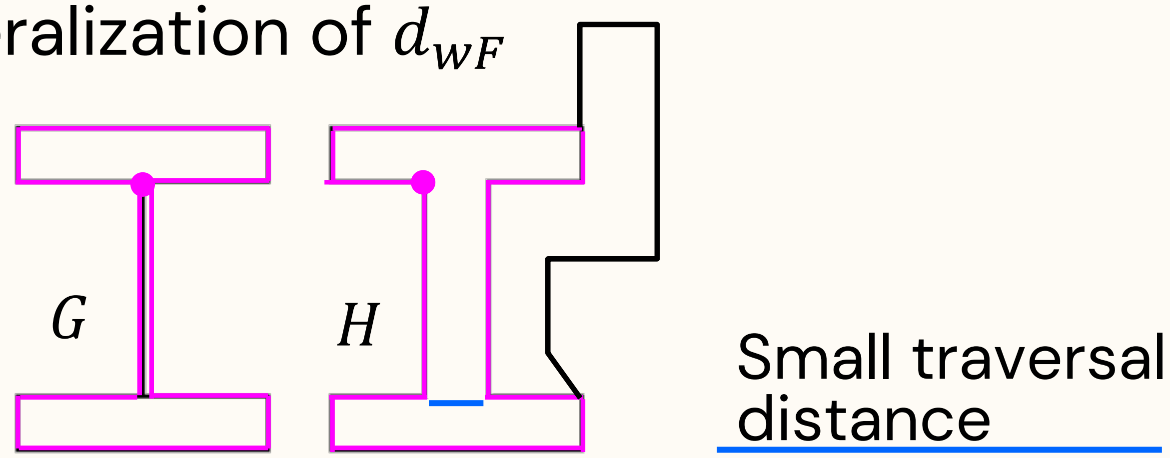
Traversal distance [AERW03]

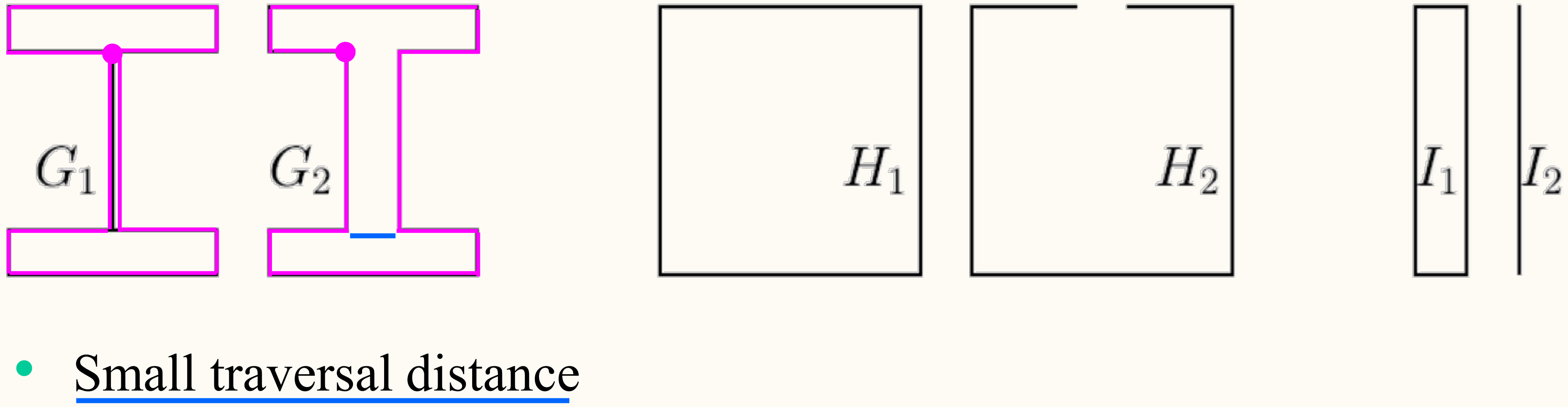
$d_T(G,H) = \inf_{g,h} \max_{t\in[0,1]} d(g(t),h(t))$, where

g: $[0,1] \to G$ are traversals (continuous, surjective),

and h: $[0,1] \to H$ partial traversals (continuous).

- Compute using free space diagram $(G \times H)$ in $O(N^2 \log N)$
- Generalization of d_{wF}





Strong (and weak) graph distances [ABKSW21]

$d_G(G,H) = \inf_{s:G\to H} \max_{e\in E_G} d_F(e,s(e))$, where

graph mappings $s: G \to H$ are continuous maps, i.e.:

- send each $v \in V_G$ to a point $s(v) \in H$, and
- send each $e \in E_G$ to a simple path from $s(u)$ to $s(v)$ in H

Weak graph distance uses d_{wF} instead.

[AERW03] H. Alt, A. Efrat, G. Rote, C. Wenk, Matching Planar Maps, Journal of Algorithms 49: 262-283, 2003.
[ABKSW21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, C. Wenk, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.
[BKK26] M. Buchin, Wolf Kißler, F. Kubon, Hardness and Parameterized Tractability of the Weak Graph Distance. WALCOM 2026: 64-78.
[BCFFGMW23] M. Buchin, E. Chambers, P. Fang, B.T. Fasy, E. Gasparovic, E. Munch, C. Wenk, Distances Between Immersed Graphs: Metric Properties, LaMatematica 2: 197-222, 2023.

Fréchet-Like Distances for Graphs

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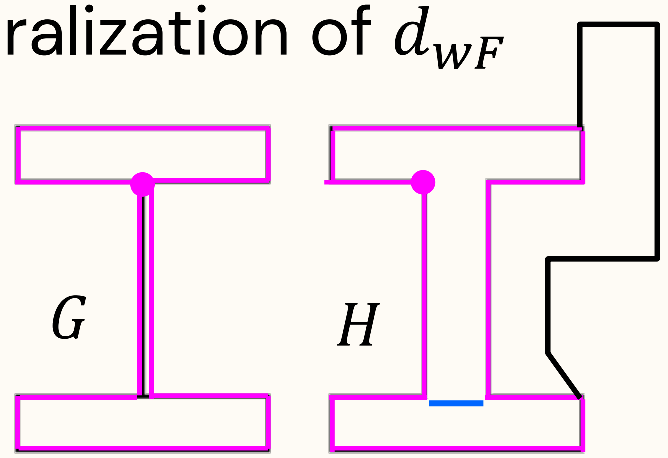
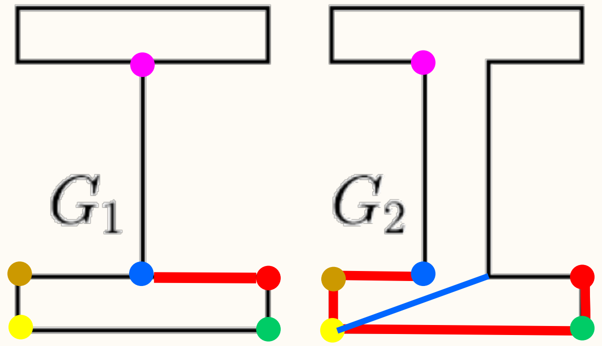
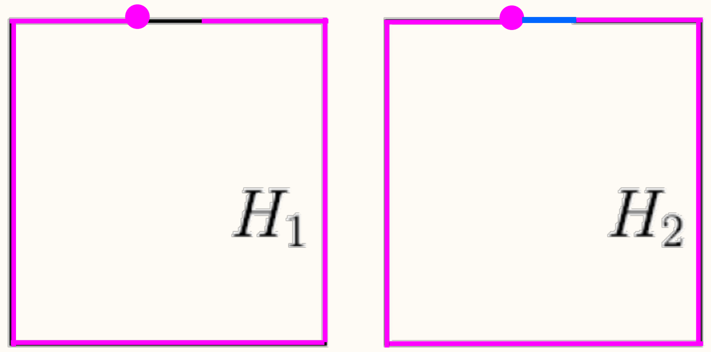
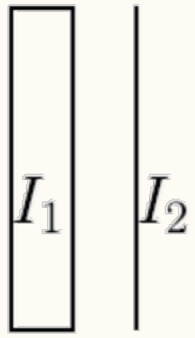


Small traversal distance



- Small traversal distance
- Large graph distance





- Small traversal distance

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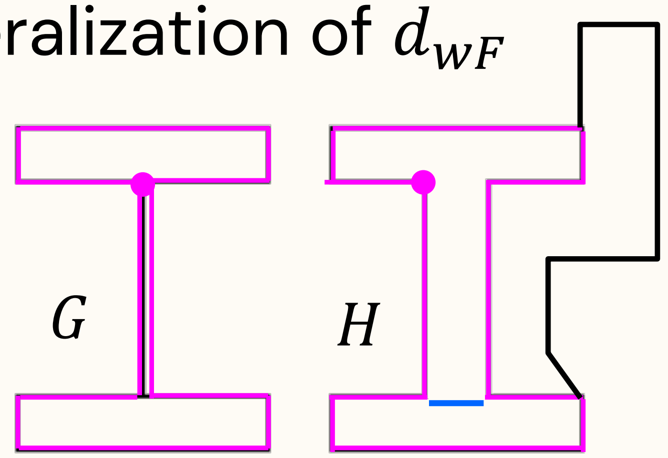
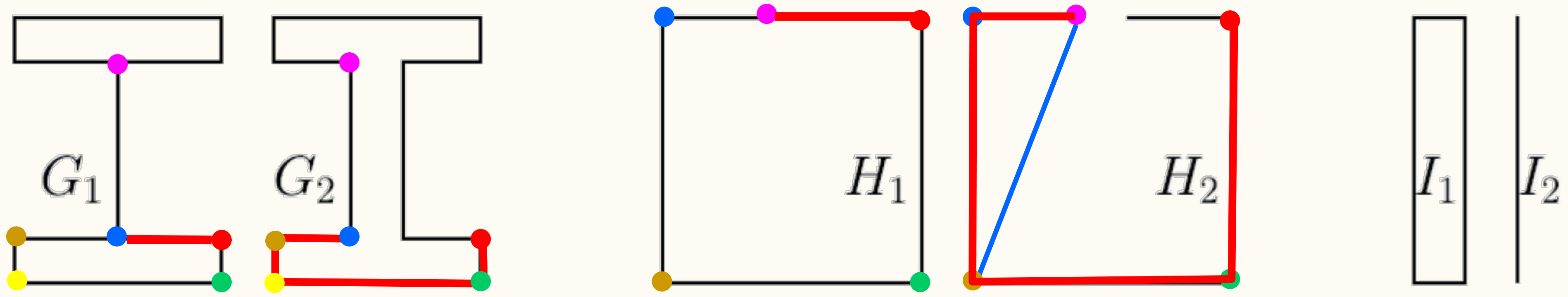


Small traversal distance



- Small traversal distance
- Large graph distance

- Small traversal distance
- Large graph distance

- Both small

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Weak graph distance uses d_{wF} instead.

- We have: $d_T(G,H) \leq d_{wG}(G,H) \leq d_G(G,H)$
- Strong distance is NP-hard, but poly time for trees.
- Weak distance is poly time for spike-free plane graphs.

[AERW03] H. Alt, A. Efrat, G. Rote, C. Wenk, Matching Planar Maps, Journal of Algorithms 49: 262-283, 2003.
[ABKSW21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, C. Wenk, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.
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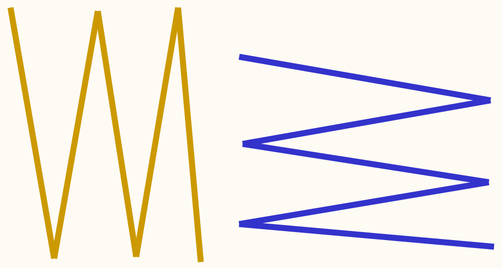
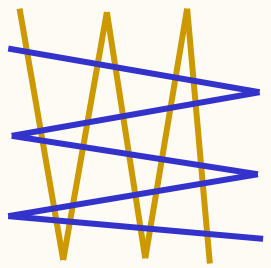
3. **Shape Matching under Transformations**
   Configuration space approaches, approximations, …

Shape Matching

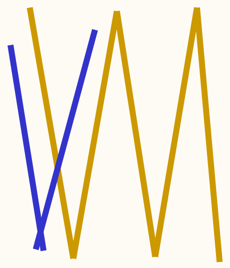
How Similar are Two Shapes?



Transformations (translations, rotations)



Choice of distance measure



Full or partial matching

Shape Matching

Let $\mathcal{T}$ be a set (or group) of transformations. E.g.:

- Translations
- rigid motions (= translations & rotations)
- isometries (= rigid motions and reflections)

Let $\mathcal{S}$ be a set of shapes.
Let d be a metric or distance on $\mathcal{S}$.

Then given two shapes $\mathrm{A}, \mathrm{B} \in \mathcal{S}$, the **shape matching problem** is to find an optimal transformation $\mathrm{T}^* \in \mathcal{T}$ such that:

$$d(T^*(A), B) = \min_{T \in \mathcal{T}} d(T(A), B)$$

Shape Matching Approaches

- **Exact:** Exhaustively search the space of all transformations; often via decision problem.
- **Approximate:** Search a subspace of transformations; exploit properties distance or shapes to provide an approximation guarantee.
  - E.g.: Reference points
- **Heuristic:** Search a subspace of transformations; no guarantee.
  - E.g.: ICP
  - E.g.: Compute shape signatures and use those to match.
- **Invariant distance design:** Avoid the shape matching framework by designing distance measures that are invariant to transformations.
  - E.g.: Compare intrinsic properties of the shapes.
  - E.g.: Gromov Hausdorff distance (NP hard to compute)

Fréchet Distance under Transformations

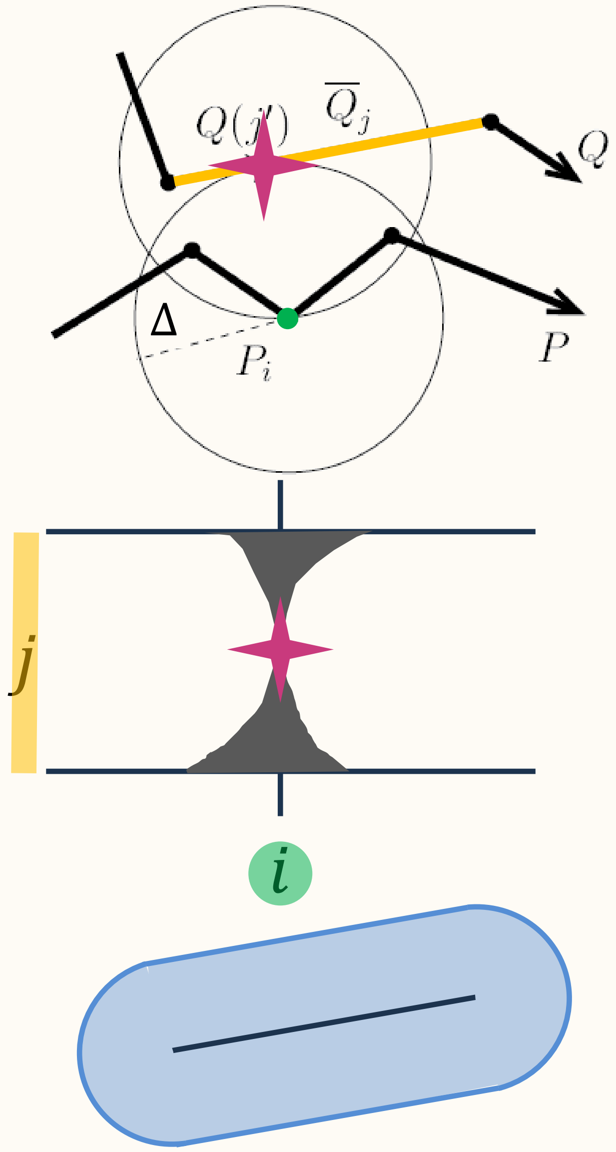
(Strong) Fréchet

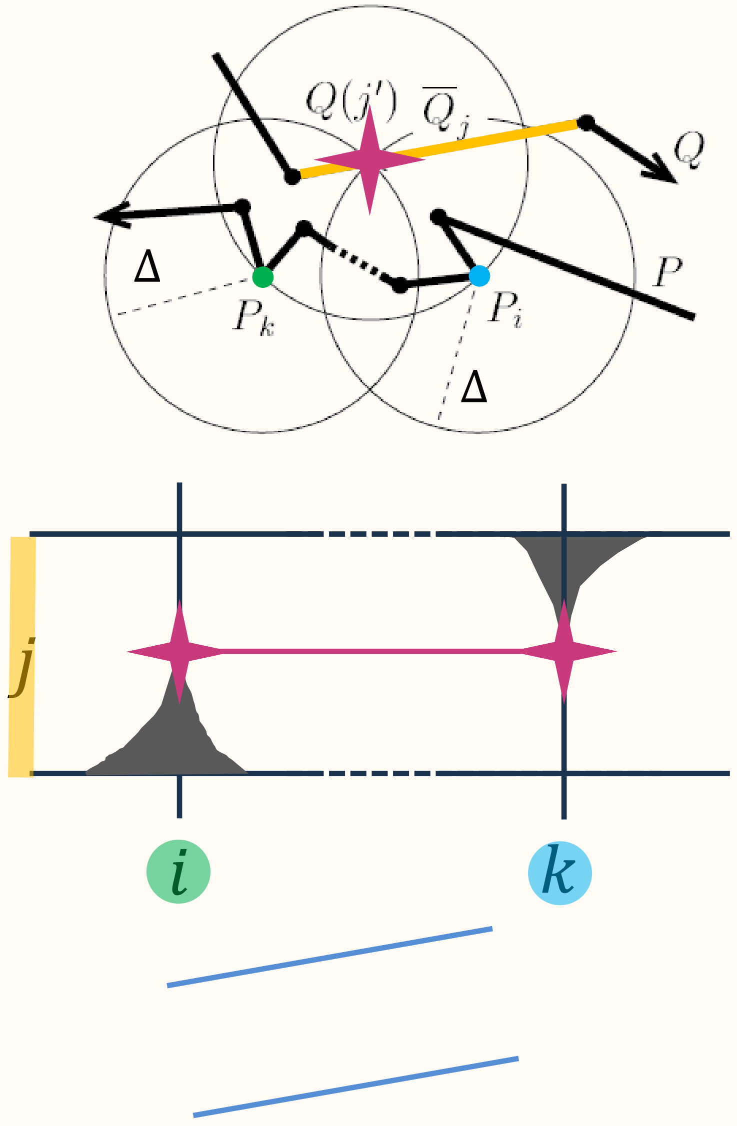
Arrangement construction

Transformations	Dim.	DOF	Time	Space
Translations	d	d	$O(N^{3d+2}\log N)$	$O(N^{3d})$
Transformations	d	k	$O(N^{3k+2}\log N)$	$O(N^{3k})$
Affine Transf.	d	d^2+d	$O(N^{3(d^2+d)+2}\log N)$	$O(N^{3(d^2+d)})$
Scalings	d	1	$O(N^5\log N)$	$O(N^3)$
Rotations	2	1	$O(N^5\log N)$	$O(N^3)$
Rotations	3	3	$O(N^{11}\log N)$	$O(N^9)$
Rigid Motions	2	3	$O(N^{11}\log N)$	$O(N^9)$
Rigid Motions	3	6	$O(N^{20}\log N)$	$O(N^{18})$
Similarities	2	4	$O(N^{14}\log N)$	$O(N^{12})$
Similarities	3	7	$O(N^{23}\log N)$	$O(N^{21})$

Translations 2 2 $O(N^8\log N) = O(N^{3\cdot 2+2}\log N)$ [AKW03]

Weak Fréchet or discrete Fréchet:
$O(N^{2k+2}\log N)$





Configuration space approach: Answer the **decision problem for given Δ** by constructing the set of valid transformations.

[W02] C. Wenk, Shape Matching in Higher Dimensions, PhD thesis, Freie Universität Berlin, 2002.
[AKW03] H. Alt, C. Knauer and C. Wenk. Comparison of distance measures for planar curves. Algorithmica, 2003.
[BKN21] K. Bringmann, M. Künnemann, A. Nusser, Discrete Fréchet Distance under Translation: Conditional Hardness and an Improved Algorithm, ACM Trans. Algorithms 17(3): 25:1–25:42, 2021
[BBHNW25] K. Buchin, M. Buchin, Z. Huang, A. Nusser, S. Wong: Faster Fréchet Distance Under Transformations. ICALP: 36:1–36:20, 2025.

Fréchet Distance under Transformations

(Strong) Fréchet

Arrangement construction

Transformations	Dim.	DOF	Time	Space
Translations	d	d	$O(N^{3d+2}\log N)$	$O(N^{3d})$
Transformations	d	k	$O(N^{3k+2}\log N)$	$O(N^{3k})$
Affine Transf.	d	d^2+d	$O(N^{3(d^2+d)+2}\log N)$	$O(N^{3(d^2+d)})$
Scalings	d	1	$O(N^5\log N)$	$O(N^3)$
Rotations	2	1	$O(N^5\log N)$	$O(N^3)$
Rotations	3	3	$O(N^{11}\log N)$	$O(N^9)$
Rigid Motions	2	3	$O(N^{11}\log N)$	$O(N^9)$
Rigid Motions	3	6	$O(N^{20}\log N)$	$O(N^{18})$
Similarities	2	4	$O(N^{14}\log N)$	$O(N^{12})$
Similarities	3	7	$O(N^{23}\log N)$	$O(N^{21})$
Translations	2	2	$O(N^8\log N) = O(N^{3\cdot 2+2}\log N)$ [AKW03]	
Translations	d	k	$O(N^{3k+4/3}\log^2 N)$	$O(N^{3k})$ [BBHNW25]

Weak Fréchet or discrete Fréchet:
$O(N^{2k+2}\log N)$

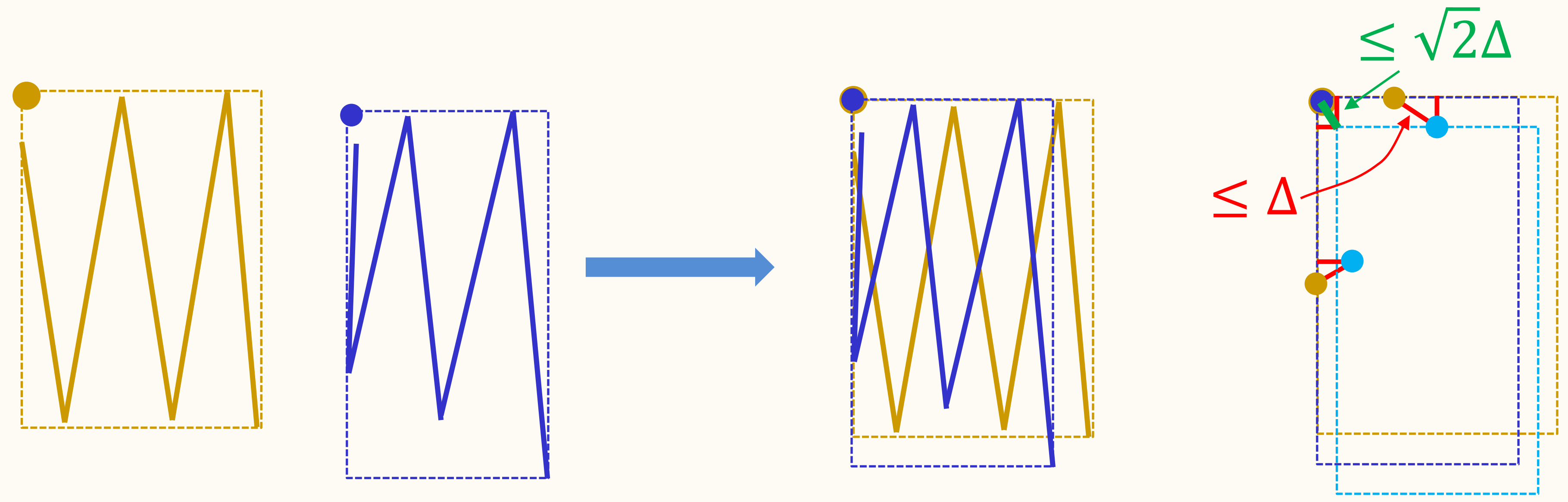
Offline dynamic grid reachability

Discrete Fréchet:
$\tilde{O}(N^{4+2/3})$, $\Omega(n^{4-o(1)})$ cond.

[W02] C. Wenk, Shape Matching in Higher Dimensions, PhD thesis, Freie Universität Berlin, 2002.
[AKW03] H. Alt, C. Knauer and C. Wenk. Comparison of distance measures for planar curves. Algorithmica, 2003.
[BKN21] K. Bringmann, M. Künnemann, A. Nusser, Discrete Fréchet Distance under Translation: Conditional Hardness and an Improved Algorithm, ACM Trans. Algorithms 17(3): 25:1-25:42, 2021
[BBHNW25] K. Buchin, M. Buchin, Z. Huang, A. Nusser, S. Wong: Faster Fréchet Distance Under Transformations. ICALP: 36:1-36:20, 2025.

Approximate Matching

Reference Points [ABB95, AAR97]



- Approximation factor of $1 + \sqrt{2} \approx 2.41$ for 2D translations under Hausdorff and Fréchet distances. $O(N)$ time.
- Steiner point: Weighted centroid of vertices of convex hull gives approx factor $1 + 4/\pi \approx 2.27$ for 2D translations and rigid motions for Hausdorff and Fréchet distances. Runtime $O(N \log N)$ for translations, but higher for rigid motions.
- Reference point: A map $r: \mathcal{S} \to \mathbb{R}^D$ that is:
  - Equivariant: $r\big(T(S)\big) = T(r(S))$ for all $A \in \mathcal{S}, T \in \mathcal{T}$
  - Lipschitz: $\big||r(A) - r(B)|\big| \leq c \cdot d(A, B)$ for all $\mathrm{A}, \mathrm{B} \in \mathcal{S}$

Pinning (exploit discrete matches)

- **Fréchet for curves:** Translating start points onto each other yields approx factor 2.
- **Hausdorff for point sets:** Fix a point in A and try out all translations that pin that point to a point in B. Runtime $O(N^2 \log N)$. [GMO99]
- Also works for discrete Fréchet, since points have to be mapped to points.

Grid-Sampling Refinements

1. Localize opt translation near pinned/ref-point alignment
2. Sample translations on a fine grid
3. **PTAS:** $O(\varepsilon\Delta)$-grid in Δ-balls around pinned translations. $\Rightarrow (1+\varepsilon)$-approximation
4. **Δ-Decision:** $O(\gamma)$-grid in Δ-ball around ref-point alignment [CKS94] $\Rightarrow$ Indecision interval $[\Delta - \gamma, \Delta + \gamma]$

[ABB95] H. Alt, B. Behrends, J. Blömer. Approximate matching of polygonal shapes. Ann. Math. Artif. Intell. 13:251–266, 1995
[AAR97] Alt, Aichholzer, Rote. Matching shapes with a reference point. IJCGA 7:349–363, 1997.
[GMO99] Goodrich, Mitchell & Orletsky. Approximate geometric pattern matching under rigid motions. PAMI 1999, 371–379.
[AKW03] H. Alt, C. Knauer and C. Wenk. Comparison of distance measures for planar curves. Algorithmica, 2003.
[CKS94] L. P. Chew, K. Kedem, S. Schirra , On characteristic points and approximate decision algorithms for the minimum Hausdorff distance. Tech report MPI-I-94-150,, 1994

(Approximate) Matching for Curves and Graphs

Approximate Matching for Curves / Fréchet

- Trivial approximation for translations:
  - **Pinning start points** of curves onto each other gives a 2-approximation in $O(1)$ time
  - **Grid-sampling** gives a $(1+\varepsilon)$-approximation in $O\left(\frac{1}{\varepsilon^2} N^2 \log N\right)$ time
- **Reference points for Fréchet:**
  - Ref points for d_H are automatically reference points for d_F, since $d_H \leq d_F$.
  - Resulting approx algos fix the translation but still optimize over the remaining transformation parameters:
    - 2D rigid motions: $O(N^{4.33} \log^2 N)$ for optimizing a 1-DOF rotation after pinning start points

Lower Bounds

- On the complexity of the configuration space:
  - $\Omega(N^2)$ for Fréchet under translations in 2D [W02]
  - Various bounds for Hausdorff, e.g., $\Omega(N^4)$ for line segments under translations in 2D [R96]
- Conditional lower bound for discrete Fréchet distance, 2D, translations $\Omega(N^{4-o(1)})$ [BKN21]

Graphs: Open

- Geometric graph distances are fairly new.
- There are no exact or approximate algorithms for matching under transformations yet.
- Some of the configuration-space approaches from Fréchet matching probably carry over to Fréchet-like graph distances, such as the traversal distance.
- Standard reference points don't work since traversal distance and the strong & weak graph distances are directed distances.

[R96] W. J. Rucklidge. Lower bounds…, DCG 16:135–153, 1996.
[W02] C. Wenk, Shape Matching in Higher Dimensions, PhD thesis, Freie Universität Berlin, 2002.
[BKN21] K. Bringmann, M. Künnemann, A. Nusser, Discrete Fréchet Distance under Translation: Conditional Hardness and an Improved Algorithm, ACM Trans. Algorithms 17(3): 25:1-25:42, 2021

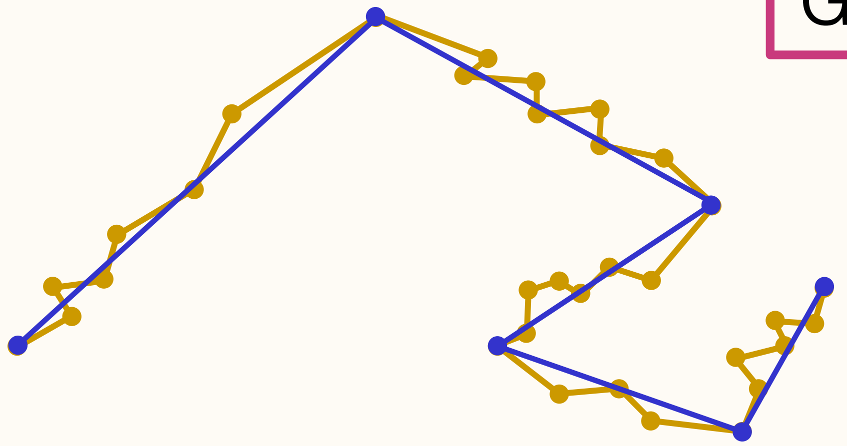
4. **Shape Simplification**
   Local vs global simplification, …

Curve Simplification

Shape Simplification

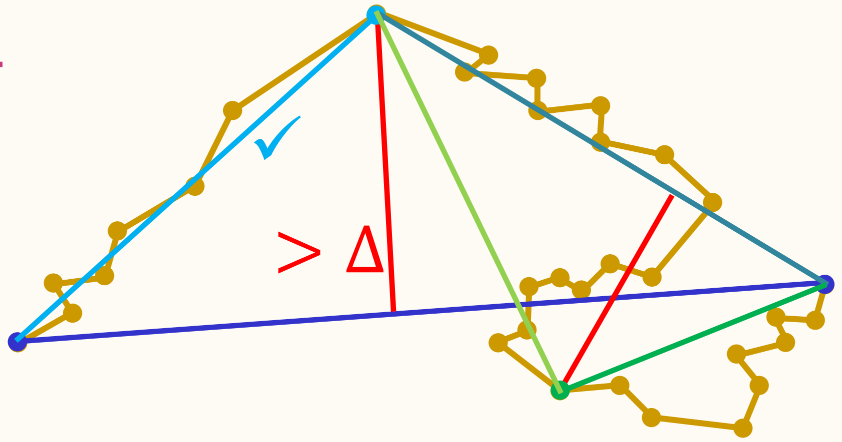
Given a shape $A \in \mathcal{S}$, find a shape
both $|B| = k$ and $d(A, B) = \Delta$ are sm

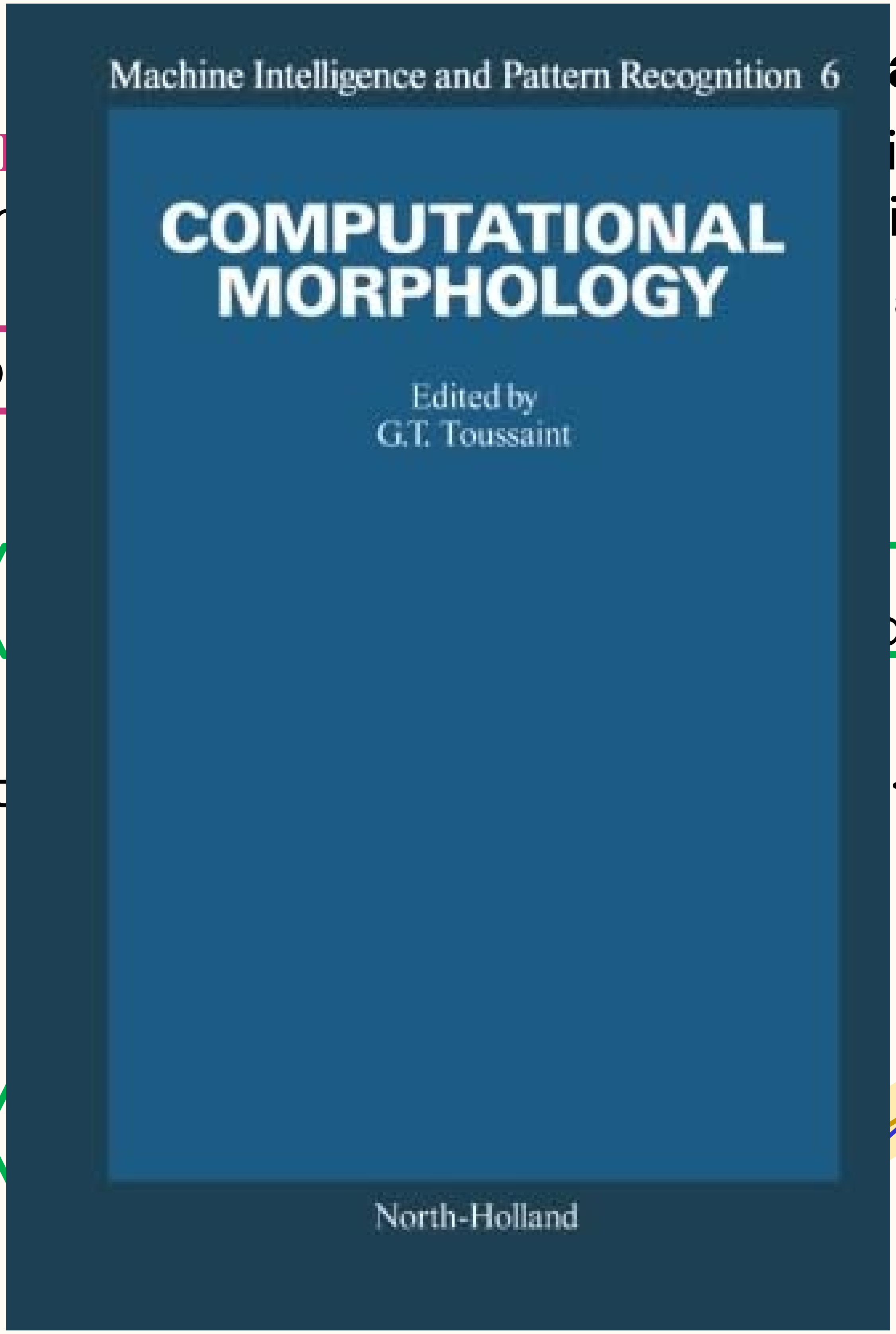
Glo



- Shortcuts between pairs of vert
- Ramer-Douglas-Peucker

Δ —

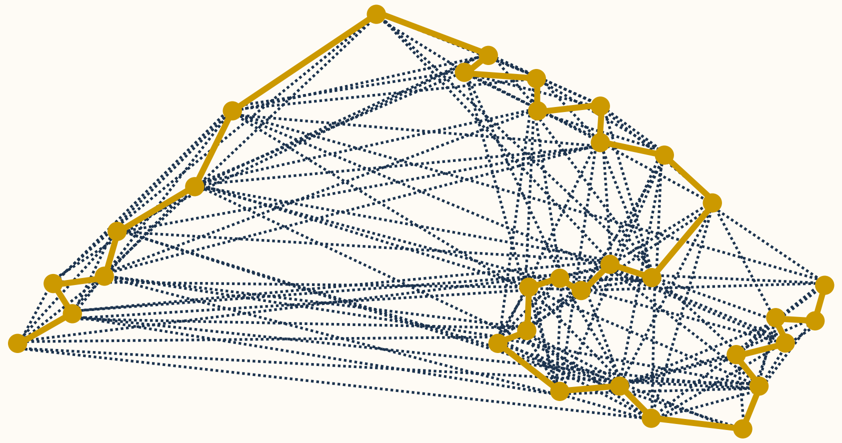




ants

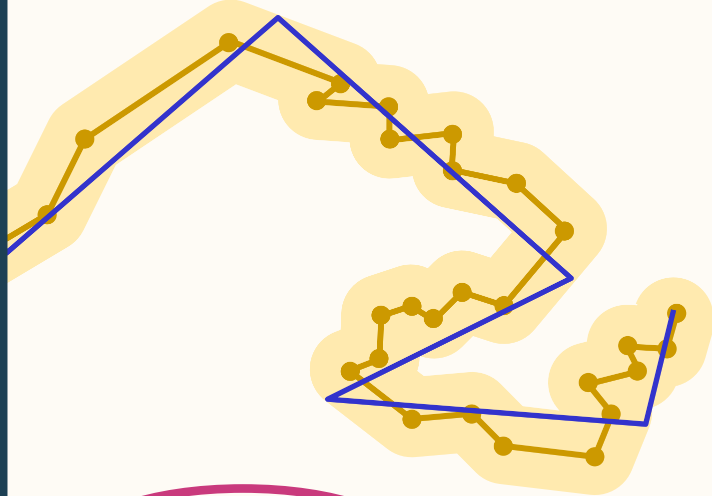
iven $\Delta > 0$, minimize $k = |B|$ s.t. $d(A, B) \leq \Delta$
iven $k > 0$, minimize $\Delta = d(A, B)$ s.t. $|B| \leq k$
graphs [II88, CC96]

d



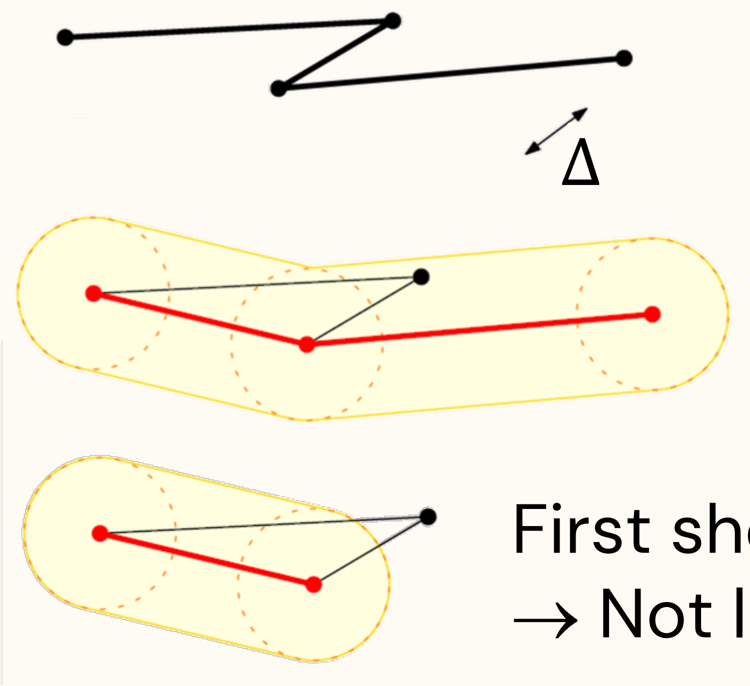
. global simplification

in Δ-neighborhood [GHMS93]



[II88] H. Imai and M. Iri, "Polygonal Approximations of a Curve: Formulations and Algorithms," in *Computational Morphology*, G. T. Toussaint, ed., North-Holland, 71-86, 1988.
[CC96] W. S. Chan and F. Chin, "Approximation of Polygonal Curves with Minimum Number of Line Segments or Minimum Error," Int. J. Comput. Geom. Appl. 6(1):*59-77,* 1996.
[GHMS93] L. J. Guibas, J. Hershberger, J.S.B. Mitchell, J. Snoeyink, "Approximating Polygons and Subdivisions with Minimum Link Paths". Int. J. Comput. Geom. Appl. 3(4): 383-415, 1993.

Global Curve Simplification



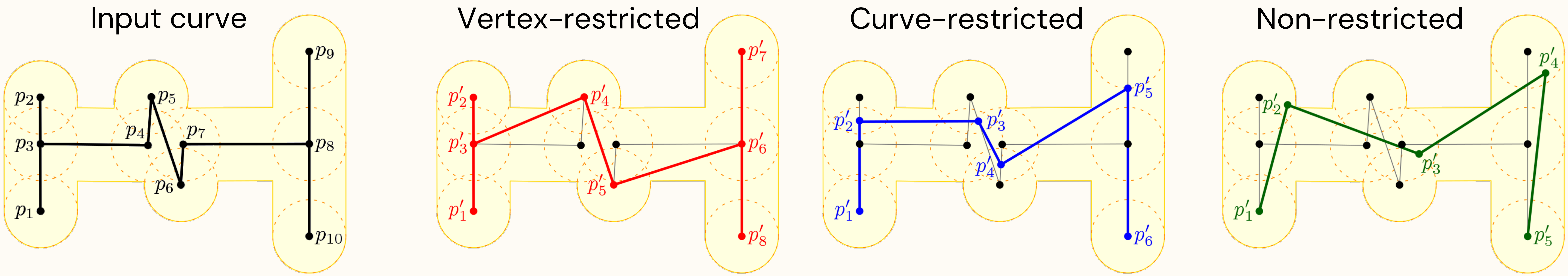
Input curve

Global simplification
(Hausdorff or Fréchet)

First shortcut, larger distance
→ Not local simplification

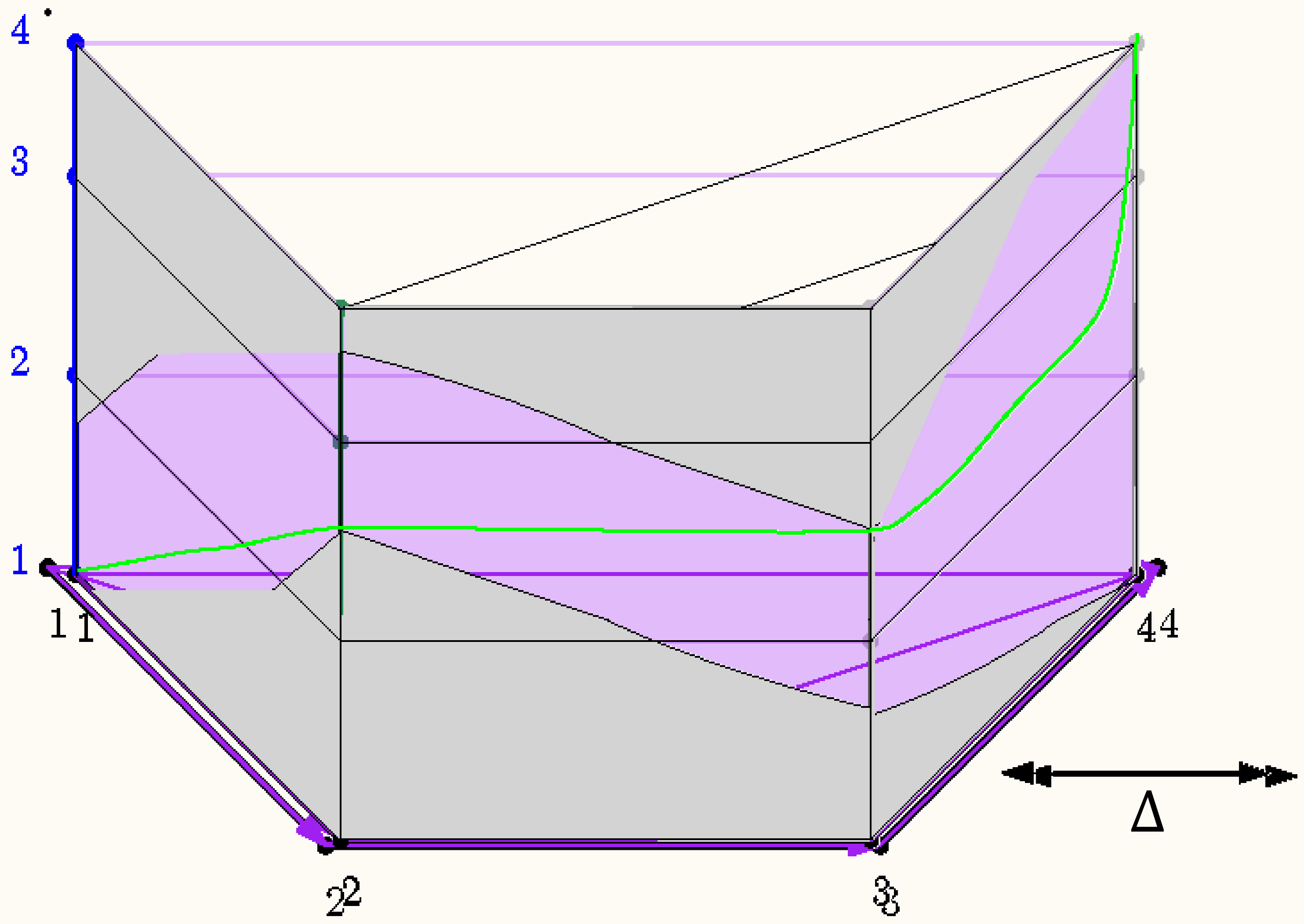
Distance	Vertex-Restr.	Curve-Restr.	Non-Restr.
$\overrightarrow{d_H}(P,\Delta)$	NP-hard [KLW18]	wNP-hard [KKLMW19]	NP-hard [KKLMW19]
$\overleftarrow{d_H}(P,\Delta)$	$O(n^3 \log n)$ [KKLMW19]	wNP-hard [KKLMW19]	poly(n) [KLPS17]
$d_H(P,\Delta)$	NP-hard [KLW18]	NP-hard [KKLMW19]	NP-hard [KKLMW19]
$d_F(P,\Delta)$	$\boxed{O(n^3)}$, $\widetilde{\Omega}(n^{3-\varepsilon})$ [KKLMW19] [BC20]	$O(n)$ in $\mathbb{R}$, wNP-hard in $\mathbb{R}^2$ [KKLMW19]	$O(n^2 \log^2 n)$ [GHMS93]
$d_{dF}(P,\Delta)$	$O(n^2)$ [BJWYZ08]	$O(n^3)$ [KKLMW19]	$O(n \log n)$ [BJWYZ08]
$d_{wF}(P,\Delta)$		wNP-hard [KKLMW19]	$(2{,}1+\varepsilon)$-approx. [KKLMW19]

Global Hausdorff simplifications



[BC20] K. Bringmann, B.R. Chaudhury, "Polyline simplification....", J. Com. Geom. 11(2): 94–130, 2020
[KKLMW19] M. vdKerkhof, I. Kostitsyna, M. Löffler, M. Mirzanezhad, C. Wenk, Global Curve...., ESA19, arXiv:1809.10269
[KLW18] M. van Kreveld, M. Löffler, and L. Wiratma. On optimal polyline simplification.... SoCG: 1-14, 2018
[KLPS17] I. Kostitsyna, M. Löffler, V. Polishchuk, F. Staals. On the complexity of"...", JoCG 8(2): 80–108, 2017
[GHMS93] L. J. Guibas, J. Hershberger, J.S.B. Mitchell, J. Snoeyink, "Approximating Polygons...". IJCGA 3(4): 383-415, 1993
[BJWYZ08] S. Bereg, M. Jiang, W. Wang, B. Yang, B. Zhu. Simplifying 3D polygonal chains...", LATIN: 630–641, 2008.

Vertex-Restricted, Global Fréchet Distance

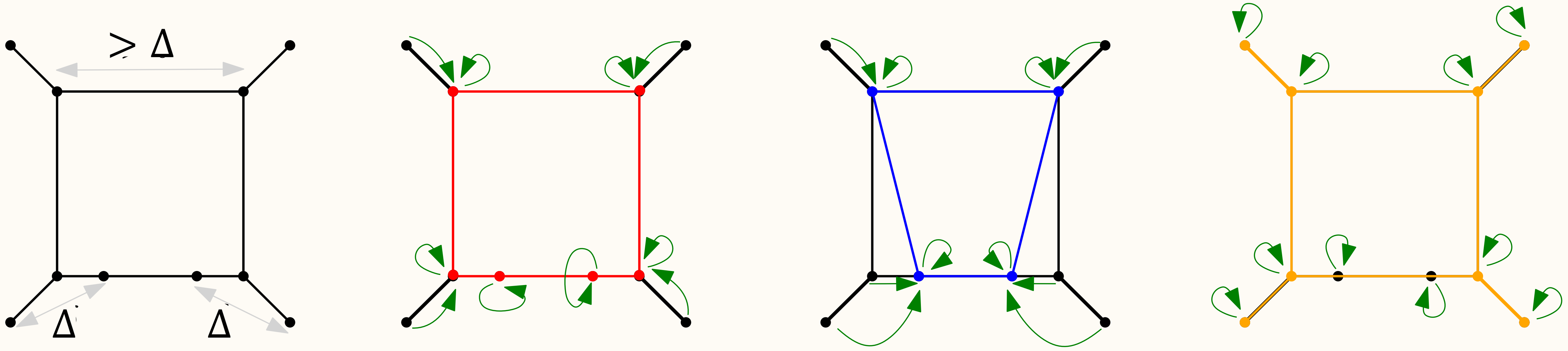


[KKLMW19] M. v.d. Kerkhof, I. Kostitsyna, M. Löffler, M. Mirzanezhad, C. Wenk, “Global Curve….”, ESA, 2019

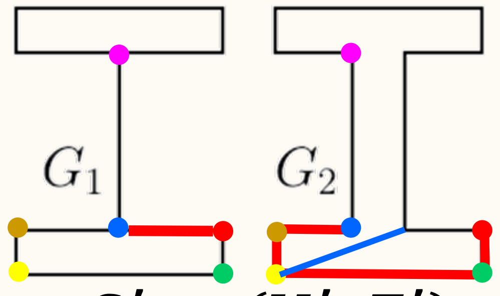
Graph Simplification

Some Hardness Results

- Vertex-restricted, local simplification; only remove deg-2 vertices
  $\rightarrow$ NP-hard to approximate within a factor $n^{1/5-\varepsilon}$ of opt [EM01]
- Homeomorphic simplification within a given polygon (e.g., Δ-neighborhood)
  $\rightarrow$ NP-hard [GHMS93]



Input graph subgraph-restricted vertex-restricted leaf-restricted



[FMW25]

Using Strong Graph Distance

Simplify geometric graph $G = (V, E)$ to $G' = (V', E')$

- Vertex-restricted: $V' \subseteq V$
  - Subgraph-restricted: G' is a subgraph of G
  - Leaf-restricted: Leaves of G' are identical to a given subset of leaves of G.
    $\rightarrow$ Generalization of start- and endpoints of curves

[FMW25] **Restrictions**

Type	**Vertex**	**Subgraph**	**Leaf**	**Edge**
Graph		NP-hard		wNP-hard *
Tree	FPT *w		NP-hard *	wNP-hard *
Tree; $\overleftarrow{d_G}$			$O(lN^5)$	

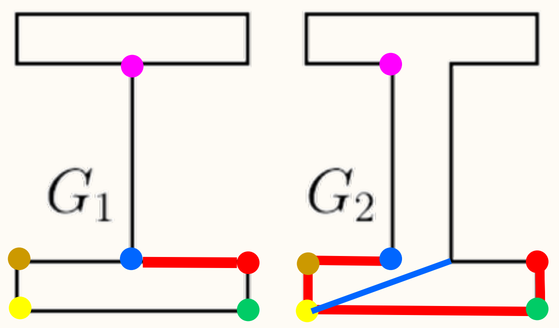
*: (weak) NP-hardness for traversal distance [M21]

[FMW25] O. Filtser, M. Mirzanezhad, C. Wenk, "Min-Complexity Graph Simplification under Fréchet-Like Distance“, IWOCA: 44-57, 2025.
[M21] M. Mirzanezhad, “Geometric Algorithms and Data Structures for Curves and Graphs”, Dissertation, Tulane University, 2021.
[EM01] R. Estkowski, J.S.B. Mitchell, Simplifying a polygonal subdivision while keeping it simple, SoCG.40–49, 2001
[GHMS93] L. J. Guibas, J. Hershberger, J.S.B. Mitchell, J. Snoeyink, “Approximating Polygons…”. IJCGA 3(4): 383-415, 1993

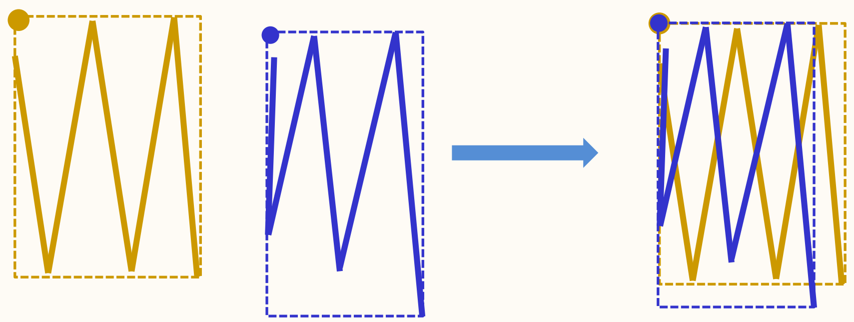
Summary / Open Problems

We discussed:

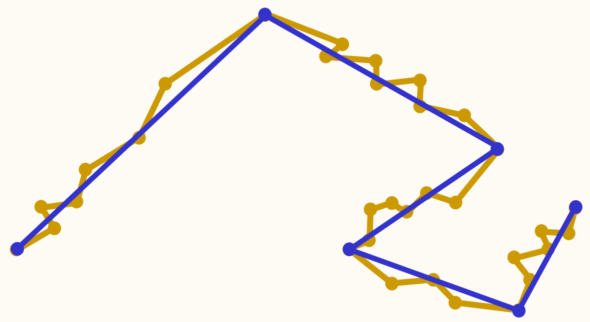
- Distances for geometric graphs



- Matching under transformations; exact and approximate



- Simplification



Open problems for matching under transformations

- Exact or approximate algorithms for graph matching; any reference points?
- Tighter lower bounds, even for curve matching. (Best known $\Omega(N^2)$ for conf space for Fréchet for translations, and $\Omega(N^{4-o(1)})$ cond. for discrete Fréchet)
- Faster matching algorithms for special curve classes, e.g., for c-packed curves?

Open problems for (global) graph simplification

- Lots of open cases. E.g., vertex-restricted for graphs and even for trees (under graph distance).
- What scenarios would enable polynomial-time algorithms?