Introduction

- Proto-Quipper-M is a programming language proposed by Rios and Selinger.
- It describes families of morphisms of an arbitrary symmetric monoidal category.
- Its categorical model is sound, but lacks support for recursion.
- Our goal is to address this limitation.

Families construction

It’s well-known that given a category $C$, we may define $\text{Fam}[C]$:
- Objects are pairs $(X, A)$ where $X$ is a discrete category and $A : X \to C$ is a functor.
- A morphism $(X, A) \to (Y, B)$ is a pair $(f, \phi)$ where $f : X \to Y$ is a functor and $\phi : A \to B \circ f$ is a natural transformation.
- Composition of morphisms is given by:
  $$(g, \psi) \circ (f, \phi) = (g \circ f, \psi \circ \phi).$$
- $\text{Fam}[C]$ is the free coproduct completion of $C$.

(Rios & Selinger 2017) If $C$ is symmetric monoidal and product-complete, then $\text{Fam}[C]$ is symmetric monoidal closed.

Categorical Model of Proto-Quipper-M

- A symmetric monoidal closed and product-complete category $M$.
- A full and faithful embedding $Q \to M$.
- A symmetric monoidal adjunction:
  $$\text{Set} \dashv \text{Fam}[M] \dashv \text{DCPO}.$$ 
- For any symmetric monoidal category $Q$, setting $M := [Q^{op}, \text{Set}]$ satisfies the first two requirements.
- Open problem: Find a categorical model which supports recursion.

Our approach

- Consider dcpo-indexed families of objects of $M$ instead of set-indexed ones.
  - Described by our $\text{DFam}$ construction.
  - Establish a new symmetric monoidal adjunction which is $\text{DCPO}$-enriched.
  - This serves as an ordered-enriched generalisation of the original model.
  - Lift the adjunction to obtain a model which supports recursion.
  - Given by the adjunction between $\text{DCPO}_{\leq}$ and $\text{DFam}_{\leq}$.

Directed Families Construction

For a given category $C$, we define $\text{DFam}[C]$ to be the category such that:
- Objects are pairs $(X, A)$ where $X$ is a poset category whose induced poset is a dcpo and $A : X \to C$ is a functor.
- A morphism $(X, A) \to (Y, B)$ is a pair $(f, \phi)$ where $f : X \to Y$ is a functor whose induced order-preserving function is Scott-continuous and $\phi : A \to B \circ f$ is a natural transformation.
- Composition of morphisms is given by:
  $$(g, \psi) \circ (f, \phi) = (g \circ f, \psi \circ \phi).$$

Properties of $\text{DFam}$

- Proposition: $\text{DFam}[C]$ has all small coproducts.
- Theorem: $\text{DFam}[C] \to \text{DCPO}$ is $\text{DCPO}_{\leq}$-enriched with order $\leq$ on $\text{DFam}_{\leq}[C]$.
  $$\text{DCPO}_{\leq} \to \text{DFam}[C]((X, A), (Y, B)) = (f, \phi) \leq (g, \psi) \iff f \leq g \text{ and } B(f) \leq g.$$ 
- Theorem: If $C$ is symmetric monoidal adjunctional and product-complete category, then $\text{DFam}[C]$ is symmetric monoidal closed. If $C$ is also complete and cocomplete, then $\text{DFam}[C]$ is also complete and cocomplete.

A DCPO-enriched model supporting recursion

- Let $Q$ be an arbitrary symmetric monoidal category (circuit model).
- Let $M$ be SMCC, product-complete, cocomplete with full and faithful embedding $Q \to M$ (e.g. $M := [Q^{op}, \text{Set}]$).
- Let $M_0$ be SMCC, product-complete, cocomplete and have a zero object with faithful embedding $M \to M_0$ (e.g. $M_0 := 1/M$).

Directed Families Construction

- For a given category $C$, we define $\text{DFam}[C]$ to be the category such that:
  - Objects are pairs $(X, A)$ where $X$ is a poset category whose induced poset is a dcpo and $A : X \to C$ is a functor.
  - A morphism $(X, A) \to (Y, B)$ is a pair $(f, \phi)$ where $f : X \to Y$ is a functor whose induced order-preserving function is Scott-continuous and $\phi : A \to B \circ f$ is a natural transformation.
  - Composition of morphisms is given by:
    $$(g, \psi) \circ (f, \phi) = (g \circ f, \psi \circ \phi).$$

Properties of $\text{DFam}$

- Proposition: $\text{DFam}[C]$ has all small coproducts.
- Theorem: $\text{DFam}[C] \to \text{DCPO}$ is $\text{DCPO}_{\leq}$-enriched with order $\leq$ on $\text{DFam}_{\leq}[C]$.
  $$\text{DCPO}_{\leq} \to \text{DFam}[C]((X, A), (Y, B)) = (f, \phi) \leq (g, \psi) \iff f \leq g \text{ and } B(f) \leq g.$$ 
- Theorem: If $C$ is symmetric monoidal adjunctional and product-complete category, then $\text{DFam}[C]$ is symmetric monoidal closed. If $C$ is also complete and cocomplete, then $\text{DFam}[C]$ is also complete and cocomplete.

- Proposition: $\text{DFam}[C]$ has a zero object and all small coproducts.
- Theorem: $\text{DFam}_{\leq}[C]$ is $\text{DCPO}_{\leq}$-enriched with order $\leq$ on $\text{DFam}_{\leq}[C]$.
  $$\text{DCPO}_{\leq} \to \text{DFam}_{\leq}[C]((X, A), (Y, B)) = (f, \phi) \leq (g, \psi) \iff f \leq g \text{ and } B(f) \leq g.$$ 
- Theorem: If $C$ is complete and has a zero object, then $\text{DFam}_{\leq}[C]$ is algebraically compact for $\text{DCPO}$-endofunctors.
- Theorem: If $C$ is symmetric monoidal closed, product-complete and has a zero object, then $\text{DFam}_{\leq}[C]$ is symmetric monoidal closed. If $C$ is also complete and cocomplete, then $\text{DFam}_{\leq}[C]$ is complete and cocomplete.

A DCPO-enriched categorical model of Proto-Quipper-M

- The category $\text{DCPO}$ is the category of classical values.
- The category $\text{DFam}[M]$ is the category of values.
- The category $\text{DFam}_{\leq}[M]$ is the category of classical computations.
- The category $\text{DFam}_{\leq}[M_0]$ is the category of computations.

Total and Partial Functions

- For a category $C$ with zero object $0$, we define the category $\text{DFam}_{\leq}[C]$ as the subcategory of $\text{DFam}[C]$.
- Objects are pairs $(X, A)$ where $X$ is a poset category whose induced poset is a pointed dcpo and $A : X \to C$ is a functor, such that $A(\bot) = 0$.
- A morphism $(X, A) \to (Y, B)$ is a pair $(f, \phi)$ where $f : X \to Y$ is a functor whose induced order-preserving function is strict Scott-continuous.

Properties of $\text{DFam}_{\leq}$

- Proposition: $\text{DFam}_{\leq}[C]$ has a zero object and all small coproducts.
- Theorem: $\text{DFam}_{\leq}[C]$ is $\text{DCPO}_{\leq}$-enriched with order $\leq$ on $\text{DFam}_{\leq}[C]$.
  $$\text{DCPO}_{\leq} \to \text{DFam}_{\leq}[C]((X, A), (Y, B)) = (f, \phi) \leq (g, \psi) \iff f \leq g \text{ and } B(f) \leq g.$$ 
- Theorem: If $C$ is complete and has a zero object, then $\text{DFam}_{\leq}[C]$ is algebraically compact for $\text{DCPO}$-endofunctors.
- Theorem: If $C$ is symmetric monoidal closed, product-complete and has a zero object, then $\text{DFam}_{\leq}[C]$ is symmetric monoidal closed. If $C$ is also complete and cocomplete, then $\text{DFam}_{\leq}[C]$ is complete and cocomplete.

A categorical model with recursion

- Let $M_0$ be SMCC, product-complete, cocomplete and have a zero object.
- Both adjunctions are $\text{DCPO}$-enriched symmetric monoidal and the right one is also $\text{DCPO}_{\leq}$-enriched:
  $$\text{DCPO}_{\leq} \to \text{DFam}_{\leq}[M_0] = \text{DFam}_{\leq}[M] \to \text{DFam}[M].$$ 
- Thus, this is a sound categorical model for a linear/non-linear lambda calculus with conditional branching and recursion on the term level which follows from a result of Benton & Wadler (LICS’96).

Proposed model of Proto-Quipper-M

- Let $Q$ be an arbitrary symmetric monoidal category (circuit model).
- Let $M$ be SMCC, product-complete, cocomplete with full and faithful embedding $Q \to M$ (e.g. $M := [Q^{op}, \text{Set}]$).
- Let $M_0$ be SMCC, product-complete, cocomplete and have a zero object with faithful embedding $M \to M_0$ (e.g. $M_0 := 1/M$).
- The category $\text{DFam}_{\leq}[M_0]$ contains the other categories as subcategories and via the adjunction to $\text{DCPO}$ may model recursion on the term level.
- Remark: $F$ has a right adjoint (shown above). If the (strong symmetric monoidal) embedding $M \to M_0$ has a right adjoint, then $E$ also has a (symmetric monoidal) right adjoint and the adjunctions above commute.

We gratefully acknowledge support of the MURI project “Semantics, Formal Reasoning, and Tool Support for Quantum Programming” sponsored by the U.S. Department of Defense and the U.S. Air Force Office of Scientific Research.